Section 3.3, Exercise 8. Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

Proof. Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Since S and T are nonempty and bounded, $\inf S, \inf T, \sup S$, and $\sup T$ all exist.

Since S is nonempty, there exists some element x of S. By definition of inf and sup, $\inf S \leq x$ and $x \leq \sup S$, so $\inf S \leq x \leq \sup S$, so $\inf S \leq \sup S$.

We show that $\inf T \leq \inf S$: if $x \in S$, then $x \in T$ (since $S \subseteq T$) so, by definition of inf, we have $\inf T \leq x$. This shows that $\inf T$ is less than or equal to every element of S, so $\inf T$ is a lower bound for S. But every lower bound is less than or equal to the greatest lower bound, so $\inf T \leq \inf S$.

Next, we show that $\sup S \leq \sup T$: if $x \in S$, then $x \in T$ (since $S \subseteq T$) so, by definition of sup, we have $x \leq \sup T$. This shows that $\sup T$ is greater than or equal to every element of S, so $\sup T$ is an upper bound for S. But every upper bound is greater than or equal to the least upper bound, so $\sup S \leq \sup T$.

We've shown that $\inf T \leq \inf S$ and $\inf S \leq \sup S$ and $\sup S \leq \sup T$. So, for any nonempty bounded subsets S and T of $\mathbb{R}$ with $S \subseteq T$,we have

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

□

Section 3.3, Exercise 12(a). Prove: if x and y are real numbers with $x < y$, then there are infinitely many rational numbers in the interval $[x, y]$.

Proof. Let $A(n)$ be the statement "if x and y are real numbers with $x < y$, then there are n rational numbers in the interval $[x, y]$." We prove by induction that $A(n)$ is true for all $n \in \mathbb{N}$.

First: by Theorem 3.3.13, there exists a rational number q_1 in the interval (x, y), and therefore in the interval $[x, y]$. So $A(1)$ is true.

Next: suppose $A(k)$ is true. That is, for real numbers x and y with $x < y$, there are k rational numbers $q_1, q_2, \ldots, q_k$ in the interval $[x, y]$. Assume these k numbers are arranged in increasing order. By Theorem 3.3.13, there is a rational number q_0 with $q_1 < q_0 < q_2$. Note that q_0 cannot be any of the numbers $q_1, q_2, q_3 \ldots, q_k$, since $q_1 < q_0 < q_2 < q_3 < \cdots < q_k$. This gives us $k + 1$ rational numbers in $[x, y]$. So $A(k + 1)$ follows.

So $A(1)$ is true and $A(k)$ implies $A(k + 1)$ for all $k \in \mathbb{N}$. So by induction, $A(n)$ is true for all n, meaning $[x, y]$ contains infinitely many rational numbers, for all $x, y \in \mathbb{R}$ with $x < y$. □