EXAM 2: SOME PRACTICE PROBLEMS

1. Mark each statement as true or false. Justify each answer: if the answer is "True," supply a careful proof, using definitions and theorems from your fact sheet. If the answer is "False," supply an *explicit* counterexample, and explain why it's a counterexample. (Please note: a picture is not a proof.)

(a) For any set $S \subseteq \mathbb{R}$, $\operatorname{bd} S \cap \operatorname{int} S = \emptyset$.

(b) Any nonempty, bounded subset $S \subseteq \mathbb{R}$ has a max and a min (that is, there exists $m \in \S$ such that $m \leq x$ for all $x \in S$, and there exists $M \in S$ such that $x \leq M$ for all $x \in S$).

(c) Any nonempty, bounded, closed subset $S \subseteq \mathbb{R}$ has a max and a min (that is, there exists $m \in S$ such that $m \leq x$ for all $x \in S$, and there exists $M \in S$ such that $x \leq M$ for all $x \in S$).

(d) For any nonempty subset $S \subseteq \mathbb{R}$, $S' \subseteq \operatorname{cl} S$.

(e) For any nonempty subset $S \subseteq \mathbb{R}$, $S' \subseteq \operatorname{bd} S$.

(f) For any nonempty subset $S \subseteq \mathbb{R}$, $\operatorname{bd} S \subseteq S'$.

(g) For any nonempty sets $S, T \subseteq \mathbb{R}$, $\operatorname{bd} S \cup \operatorname{bd} T = \operatorname{bd}(S \cup T)$.

(h) A nonempty subset $S \subseteq \mathbb{R}$ contains all of its interior points.

(i) For any nonempty sets $S, T \subseteq \mathbb{R}$ such that $S \subseteq T$, we have $\inf S \leq \inf T$.

(j) For any nonempty subset $S \subseteq \mathbb{R}$, each isolated point of S is also a boundary point of S.

(k) A compact set in $\mathbb{R}$ contains all of its boundary points.

(l) $x \in \mathbb{R}$ is an accumulation point of a set $S \subseteq \mathbb{R}$ iff $\forall \varepsilon > 0 : N(x, \varepsilon) \cap S \neq \emptyset$.

(m) If $x \in \operatorname{bd} S$, where S is a nonempty subset of $\mathbb{R}$, then $N^*(x, 0.01) \cap S \neq \emptyset$.

(n) S is open iff $S = \operatorname{int} S$.

(o) Every neighborhood is an open set.

(p) The union of any collection of closed sets is closed.

(q) If $s_n \to 0$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n < \varepsilon$.

(r) If for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n < \varepsilon$, then $s_n \to 0$.

(s) Given sequences (s_n) and (a_n), if, for some $s \in \mathbb{R}$, $k > 0$ and $m \in \mathbb{N}$ we have $|s_n - s| \leq k|a_n|$ for all $n > m$, then $\lim s_n = s$.

(t) If $s_n \to s$ and $s_n \to t$, then $s = t$.

(u) If $s_n \to s$ and $s_n t_n \to st$, where $s_n \neq 0$ for any $n \in \mathbb{N}$ and $s \neq 0$, then $t_n \to t$.

(v) If (s_n) and (t_n) are divergent sequences, then $(s_n + t_n)$ diverges.

(w) If (s_n) and (t_n) are divergent sequences, then $(s_n t_n)$ diverges.

(x) If (s_n) and $(s_n + t_n)$ are convergent sequences, then (t_n) converges.

2. Find the max, min, sup, and inf of each of the following sets. If the quantity in question does not exist, say so. (You don't need to prove or explain anything here.)

(a) $\{0, 1\}$

(b) $(0, 1)$

(c) $\{0 - \frac{1}{n} : n \in \mathbb{N}\}$

(d) $\bigcup_{n=1}^{\infty} \left(1 + \frac{3}{n}, 1 - \frac{3}{n}\right)$

(e) $\left\{(-1)^n \left(3 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(f) $\{r \in \mathbb{Q} : r^2 < 5\}$

(g) $\{r \in \mathbb{Q} : r^2 \leq 5\}$

3. Provide an example of a set that has infinitely many accumulation points, none of which are interior points.

4. Let S and T be subsets of $\mathbb{R}$. Find a counterexample for each of the following.

(a) If P is the set of all isolated points of S, then P is a closed set.

(b) If S is open, then $\operatorname{int}(\operatorname{cl} S)) = S$.

(c) $\operatorname{bd}(\operatorname{cl} S) = \operatorname{bd} S$.

(d) $\operatorname{bd}(S \cup T) = (\operatorname{bd} S) \cup (\operatorname{bd} T)$.

5. Fill in the blanks to prove the following.

Theorem. An accumulation point of a set S is either an interior point of S or a boundary point of S.

Proof. Let $x \in S'$; we need to conclude that $x \in \operatorname{int} S \cup$ ____________. If $x \in \operatorname{int} S$, then we're done. So suppose $x \notin \operatorname{int} S$. We must show that $x \in$ ____________. By definition of ________________ point, this means: we must show that any neighborhood $N(x, \varepsilon)$ of x intersects both S and ____________ .

So let $N(x, \varepsilon)$ be such a neighborhood. Since x is an accumulation point of S we know, by definition of accumulation point, that $N^*(x, \varepsilon)$ intersects __________ ; since $N^*(x, \varepsilon) \subseteq N(x, \varepsilon)$, we conclude that ______________ intersects S as well. So we need only show that $N(x, \varepsilon)$ intersects ____________ .

But we're assuming that $x \notin$ ____________, so no neighborhood $N(x, \varepsilon)$ can lie completely inside __________ , so $N(x, \varepsilon)$ must intersect ____________ .

So any neighborhood $N(x, \varepsilon)$ of x intersects both __________ and ____________ .

Therefore, $x \in S' \Rightarrow x \in$ ____________ $\cup$ ____________, and we're done. □ □

6. Using only the definition of limit given on the fact sheet, show *carefully* that

$$\lim_{n\to\infty} \frac{4n+3}{n+3} = 4.$$

7. Using only the definition of limit given on the fact sheet, show *carefully* that

$$\lim_{n\to\infty} \frac{n^2-n+3}{2n^2-8} = \frac{1}{2}.$$

8. Using only the definition of limit given on the fact sheet, show *carefully* that

$$\lim_{n\to\infty} \frac{n^3-n+3}{2n^2-8} = +\infty.$$

9. Using only the definition of limit given on the fact sheet, show *carefully* that

$$\lim_{n\to\infty} \frac{n^3-n+3}{8-2n^2} = -\infty.$$

10. Using any of the limit laws (item **D(xiv)** from the fact sheet), show *carefully* that

$$\lim_{n\to\infty} \frac{n^2-n+3}{2n^2-8} = \frac{1}{2}.$$

11. (a) Find the interior, boundary, accumulation points, isolated points, and closure of the set

$$A = [-5,1) \cup \left\{ 2 + \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

You don't need to justify your answers.

int A = ______________________________

bd A = ______________________________

A' = ______________________________

$A \backslash A'$ = ______________________________

cl A = ______________________________

(b) Is A open, closed, or neither? Please explain carefully, using definitions and/or theorems from your fact sheet.

(c) Is A compact? Please explain carefully, using definitions and/or theorems from your fact sheet.

12. Repeat the previous exercise for the set $[0,1]\backslash\mathbb{Q}$ of **irrational** numbers in the interval $[0,1]$.

13. Fill in in the blanks.

Theorem. The set $\mathbb{N}$ of natural numbers is not ________________ .

Proof. To show $\mathbb{N}$ is not compact, we need to find an open cover of $\mathbb{N}$ with no finite subcover. That is, we need to find a collection $\mathcal{C}$ of _____________ sets such that $\mathbb{N}$ is contained in the union of the sets in $\mathcal{C}$, but $\mathbb{N}$ is *not* contained in the union of any finite number of sets in __________ .

Let $\mathcal{C} = \{I_n \colon n \in \mathbb{N}\}$, where I_n is the open interval $I_n = (n - \frac{1}{2}, n + \frac{1}{2})$. Then each I_n is open (since I_n is an open interval), and certainly

$$\mathbb{N} \subseteq \cup_{n\in\mathbb{N}} I_n$$

because, if n is a positive integer, then n is in the interval ___________ . So $\mathcal{C}$ is an open _____________ of $\mathbb{N}$.

To show that $\mathcal{C}$ has no finite subcover of $\mathbb{N}$, consider any *finite* set of intervals of the form $(n - \frac{1}{2}, n + \frac{1}{2})$. Let's say there are K intervals in this finite set. List them in increasing order: that is, list them as

$$\left(n_1 - \frac{1}{2}, n_1 + \frac{1}{2}\right), \left(n_2 - \frac{1}{2}, n_2 + \frac{1}{2}\right), \left(n_3 - \frac{1}{2}, n_2 + \frac{1}{2}\right), \ldots, \left(n_K - \frac{1}{2}, n_K + \frac{1}{2}\right), \qquad (*)$$

where $n_1 < n_2 < n_3 < \cdots < n_K$. (It's a fact that every finite set of integers can be written in increasing order; proof omitted.) Since the integer $n_K + 1$ is larger than $n_K + \frac{1}{2}$, we see that the integer _______________ is not in any of the intervals in $(*)$, and therefore, is not in the union of these intervals.

So we've shown that every finite subcover of the open cover $\mathcal{C}$ of $\mathbb{N}$ *fails* to cover ___________ . In other words, we've found an open cover of $\mathbb{N}$ with no finite ________________ . So $\mathbb{N}$ is not ________________ . $\square$

14. Consider the real numbers $\mathbb{R}$, with the usual multiplication, denoted as usual by "$\cdot$," and with an "addition" operator "@" defined by

$$x \,@\, y = \text{ the mean (average) of } x \text{ and } y \;= \frac{x+y}{2}.$$

(a) Show that, with these definitions of addition "@" and multiplication "$\cdot$," the distributive law holds. That is, show that

$$x \cdot (y \,@\, z) = (x \cdot y) \,@\, (x \cdot z) \quad \forall x, y, z \in \mathbb{R}.$$

(b) Use an explicit counterexample to show that, with this definition of addition @, the associative law for addition, which is the statement that

$$x@(y@z) = (x@y)@z \quad \forall x, y, z \in \mathbb{R},$$

does not hold.

15. Prove that the intersection of any collection of compact sets is compact.

16. Prove that, if $s_n \to +\infty$ and $k < 0$, then $ks_n \to -\infty$.