A. The Completeness Axiom.

Recall that, for $S \subseteq \mathbb{R}$, the **supremum** of S, denoted $\sup S$, if it exists, is the least upper bound for S, and the **infemum** of S, denoted $\inf S$, if it exists, is the greatest lower bound for S.

The Completeness Axiom states: If a nonempty set $S \subseteq \mathbb{R}$ is bounded above (respectively, below), then $\sup S$ (respectively, $\inf S$) exists as a real number.

B. The definition of a limit.

(a) Let $s \in \mathbb{R}$ and let (s_n) be a sequence of real numbers. We say that **the sequence (s_n) converges to s**, and write

$$\lim_{n\to\infty} s_n = s, \qquad \text{or} \qquad \lim s_n = s, \qquad \text{or} \qquad s_n \to s,$$

if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow |s_n - s| < \varepsilon.$$

(b) A sequence (s_n) is said to diverge to $+\infty$, and we write $s_n \to +\infty$, provided that

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \ n \geq N \Rightarrow s_n > M.$$

(c) A sequence (s_n) is said to diverge to $-\infty$, and we write $s_n \to -\infty$, provided that

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} : \ n \geq N \Rightarrow s_n < M.$$

C. Some definitions concerning the topology of $\mathbb{R}$.

(i) A **neighborhood** of a point $x \in \mathbb{R}$ is a set $N(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$, for some $\varepsilon > 0$.

(iii) A deleted neighborhood of a point $x \in \mathbb{R}$ is a set $N^*(x, \varepsilon) = (x - \varepsilon, x) \cup (x, x + \varepsilon)$, for some $\varepsilon > 0$.

(iv) An interior point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for some $\varepsilon > 0$, $N(x, \varepsilon) \subseteq S$. The set of all interior points of S is denoted $\operatorname{int} S$.

(v) A boundary point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for all $\varepsilon > 0$, $N(x, \varepsilon) \cap S \neq \emptyset$ and $N(x, \varepsilon) \cap \mathbb{R} \backslash S \neq \emptyset$. The set of all boundary points of S is denoted $\operatorname{bd} S$.

(vi) A set $S \subset \mathbb{R}$ is **closed** if $\operatorname{bd} S \subseteq S$. A set $S \subseteq \mathbb{R}$ is **open** if $\operatorname{bd} S \subseteq \mathbb{R} \backslash S$.

(vii) An accumulation point of set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that, for every $\varepsilon > 0$, $N^*(x, \varepsilon) \cap S \neq \emptyset$. The set of all accumulation points of S is denoted S'.

(viii) An isolated point of a set $S \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ such that $x \in S$ but $x \notin S'$. The set of all isolated points of S is simply the set $S \backslash S'$.

(ix) A set $S \subseteq \mathbb{R}$ is called **compact** if every open cover of S (that is, every collection $\{T_\alpha \colon \alpha \in \mathcal{A}\}$ of open sets T_α whose union contains S) has a finite subcover (meaning there exists a collection of finitely many of the T_α's such that the union of these finitely many T_α's contains S).

(x) The **closure** $\operatorname{cl} S$ of a set $S \subseteq \mathbb{R}$ is defined by $\operatorname{cl} S = S \cup S'$.

D. Some theorems that you may use without proof (but you must cite the appropriate theorem at any point where it is needed).

(i) Theorem 3.1.2: The Principle of Mathematical Induction. Let $A(n)$ be a statement regarding a natural number n. Suppose that **(a)** $A(1)$ is true, and **(b)** $A(k)$ implies $A(k+1)$, for all $k \in \mathbb{N}$. Then $A(n)$ is true for all integers n.

(ii) Theorem 3.2.10(d) and Exercise 3.2.6(a): The Triangle Inequality on $\mathbb{R}$. Let $x, y \in \mathbb{R}$. Then: **(a)** $|x+y| \leq |x| + |y|$. **(b)** $\big| |x| - |y| \big| \leq |x-y|$.

(iii) Theorem 3.3.9: The Archimedean Property of $\mathbb{R}$. The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.

(iv) Theorem 3.3.10: Each of the following is equivalent to the Archimedean Property. **(a)** For each $z \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > z$. **(b)** For each $x > 0$ and for each $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx > y$. **(c)** For each $x > 0$, there exists an $n \in N$ such that $0 < 1/n < x$.

(v) Theorem 3.3.13: The Density of $\mathbb{Q}$ in $\mathbb{R}$. If x and y are real numbers with $x < y$, then there exists a rational number r with $x < y < r$.

(vi) Theorem 3.4.7. Let S be a subset of $\mathbb{R}$. **(a)** S is open iff $S = \operatorname{int} S$. **(b)** S is closed iff its complement $\mathbb{R} \backslash S$ is open.

(vii) Theorem 3.4.10 and Corollary 3.4.11. **(a)** The union of any collection of open sets is open. **(b)** The intersection of any finite collection of open sets is open. **(c)** The intersection of any collection of closed sets is closed. **(d)** The union of any finite collection of closed sets is closed.

(viii) Theorem 3.4.17. Let S be a subset of $\mathbb{R}$. **(a)** S is closed iff $S' \subseteq S$. **(b)** $\operatorname{cl} S$ is closed. **(c)** S is closed iff $S = \operatorname{cl} S$. **(d)** $\operatorname{cl} S = S \cup \operatorname{bd} S$.

(ix) Theorem 3.5.5 (Heine-Borel). A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

(x) Theorem 3.5.6 (Bolzano-Weierstrass). If $S \subseteq \mathbb{R}$ is bounded and contains infinitely many points, then there is at least one point in $\mathbb{R}$ such that $x \in S'$ (that is, such that x is an accumulation point of S).

(xi) Theorem 4.1.9. Let (s_n) and (a_n) be sequences of real numbers and let $s \in \mathbb{R}$. If for some $k > 0$ and some $m \in \mathbb{N}$ we have $|s - s_n| < k|a_n|$ for all $n \geq m$, and if $a_n \to 0$, then it follows that $s_n \to s$.

(xii) Theorem 4.1.15. Every convergent sequence is bounded.

(xiii) Theorem 4.1.16. If a sequence converges, its limit is unique.

(xiv) Theorem 4.2.1. Suppose (s_n) and (t_n) are convergent sequences with $s_n \to s$ and $t_n \to t$. Then **(a)** $s_n + t_n \to s + t$. **(b)** $k + s_n \to k + s$ and $ks_n \to ks$ for any $k \in \mathbb{R}$. **(c)** $s_n t_n \to st$. **(d)** If $t \neq 0$ and $t_n \neq 0$ for any $n \in \mathbb{N}$, then $s_n / t_n \to s/t$.

(xv) Theorem 4.2.4. Suppose (s_n) and (t_n) are convergent sequences with $s_n \to s$ and $t_n \to t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $s \leq t$.

(xvi) Corollary 4.2.5. If $t_n \to t$ and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.

(xvi) Theorem 4.2.7. Suppose that (s_n) is a sequence of positive terms and that the sequence of ratios (s_{n+1}/s_n) converges to L. If $L < 1$, then $s_n \to 0$.

(xvi) Theorem 4.2.12. Suppose (s_n) and (t_n) are sequences such that $s_n \leq t_n \, \forall n \in \mathbb{N}$. **(a)** If $s_n \to +\infty$ then $t_n \to +\infty$. **(b)** If $t_n \to -\infty$ then $s_n \to -\infty$.

(xvii) Theorem 4.2.13. (s_n) be a sequence of positive numbers. Then $s_n \to \infty$ iff $1/s_n \to 0$.

E. Quantifiers.

1. The quantifier "∀" means "for all," or "for each," or "for every."
   If X is a set and $Q(x)$ is a statement about a quantity x, then the statement
   $$\forall x \in X : Q(x)$$
   means the statement $Q(x)$ is true for every x in X.
2. The quantifier "∃" means "for some," or "for at least one," or "there exists."
   If X is a set and $Q(x)$ is a statement about a quantity x, then the statement
   $$\exists x \in X : Q(x)$$
   means the statement $Q(x)$ is true for some (at least one, possibly more) x in X.

F. Proof templates.

(a) $P \Rightarrow Q$, direct proof.

> **Theorem.** $P \Rightarrow Q$.
> **Proof.** Assume P. [Now do what you need to conclude:] Therefore, Q.
> So $P \Rightarrow Q$. □

(b) $P \Rightarrow Q$, contrapositive proof.

> **Theorem.** $P \Rightarrow Q$.
> **Proof.** Assume $\sim Q$. [Now do what you need to conclude:] Therefore, $\sim P$.
> So $P \Rightarrow Q$. □

(c) $P \Leftrightarrow Q$.

> **Theorem.** $P \Leftrightarrow Q$.
> **Proof.** Assume P. [Now do what you need to conclude:] Therefore, Q.
> So $P \Rightarrow Q$.
> Next, assume Q. [Now do what you need to conclude:] Therefore, P.
> So $Q \Rightarrow P$.
> Therefore, $P \Leftrightarrow Q$. □

(d) Proofs with universal quantifiers.

> **Theorem.** $\forall x \in X$, $Q(x)$.
> **Proof.** Assume $x \in X$. [Now do what you need to conclude:] Therefore, $Q(x)$.
> So $\forall x \in X$, $Q(x)$. □

(e) Proofs with existential quantifiers.

Theorem. $\exists x \in X$, $Q(x)$.
Proof. [Find a particular $x \in X$, call it x_0, that has the property $Q(x)$. Then write:] Let $x = x_0$. Then ... [show that $Q(x_0)$ is true]. So $\exists x \in X$, $Q(x)$. □

(f) Proof by contradiction.

Theorem. T.
Proof. Assume $\sim T$. [Then do what's necessary to derive a contradiction, and write:] Contradiction. Therefore T is true. □

(g) Proof by the principle of mathematical induction.

Theorem. $\forall n \in \mathbb{N},\ A(n)$.
Proof. Step 1: Is $A(1)$ true? [Now do what you need to conclude:] So $A(1)$ is true.
Step 2: Assume $A(k)$. [Now do what you need to conclude:] So $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.
Therefore, by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. □

G. Some special sets.

(a) $\mathbb{Z} = \{\text{integers}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

(b) $\mathbb{N} = \{\text{natural numbers}\} = \{1, 2, 3, \ldots\}$.

(c) $\mathbb{R} = \{\text{real numbers}\} = (-\infty, \infty)$.

(d) $\mathbb{Q} = \{\text{rational numbers}\} = \{\text{fractions } m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

H. Facts about integers.

(a) Let $a, b \in \mathbb{Z}$. We say a divides b, written $a|b$, if $b = na$ for some $n \in \mathbb{Z}$.

(b) (Division algorithm.) Given integers a and b with $b > 0$, there exist unique integers q and r for which $a = qb + r$ and $0 \leq r < b$.