EXAM 1: SOME PRACTICE PROBLEMS

1. Prove the following.

Proposition.

(a) If $a, m \in \mathbb{Z}$ and $m|a$, then $m|(an)$ for any $n \in \mathbb{Z}$.

(b) If $a, b, m \in \mathbb{Z}$, $m|a$, and $m|b$, then $m|(a + b)$ and $m|(a - b)$.

(Use the definition of "|" from your fact sheet.)

Proof. (a) Let $a, m \in \mathbb{Z}$; suppose $m|a$. Then $a = mc$ for some $c \in \mathbb{Z}$. But then, if $n \in \mathbb{Z}$, we have $an = mcn = m(cn)$, so $m|(an)$. Therefore, $a, m \in \mathbb{Z}$ and $m|a \Rightarrow m|(an)$ for any $n \in \mathbb{Z}$.

(b) Let $a, b, m \in \mathbb{Z}$; suppose $m|a$ and $m|b$. Then $\exists q, r \in \mathbb{Z}$ with $a = mq$ and $b = mr$. But then $a + b = mq + mr = m(q + r)$, so $m|(a + b)$. Therefore, $a, b, m \in \mathbb{Z}$, $m|a$, and $m|b \Rightarrow m|(a + b)$. The proof that $m|(a - b)$ under these assumptions is similar. $\square$

2. Prove the following.

Proposition. If $a, b, c, d, m \in \mathbb{Z}$, $m|(a - b)$, and $m|(c - d)$, then $m|\big(a + c - (b + d)\big)$. (Use the definition of "|" from your fact sheet.)

Proof. Let $a, b, c, d, m \in \mathbb{Z}$; suppose $m|(a - b)$ and $m|(c - d)$. Then $\exists q, r \in \mathbb{Z}$ with $a - b = mq$ and $c - d = mr$. But then

$$a + c - (b + d) = (a - b) + (c - d) = mq + mr = m(q + r),$$

so $m|(a + c - (b + d))$. Therefore, $a, b, c, d, m \in \mathbb{Z}$, $m|(a - b)$, and $m|(c - d) \Rightarrow m|(a + c - (b + d))$. $\square$

3. Prove the following.

Proposition. If $a, b \in \mathbb{Z}$, a is odd, and b is even, then ab is even. (Use the definitions of even and odd integers: an even integer is one of the form $2k$ for some $k \in \mathbb{Z}$; an odd integer is one of the form $2k + 1$ for some $k \in \mathbb{Z}$.)

Proof. Assume that $a, b \in \mathbb{Z}$, a is odd, and b is even. Then we can write $a = 2k$ and $b = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. But then

$$ab = (2k)(2\ell + 1) = 2 \cdot (k(2\ell + 1)),$$

so ab is 2 times an integer, so ab is even. So $a, b \in \mathbb{Z}$, a is odd, and b is even $\Rightarrow ab$ is even. $\square$

4. Use proof by contrapositive to prove the following.

Proposition. If n^2 is odd, then n is odd. (Use the definitions of even and odd integers: an even integer is one of the form $2k$ for some $k \in \mathbb{Z}$; an odd integer is one of the form $2k + 1$ for some $k \in \mathbb{Z}$.)

Proof. Suppose n is not odd. Then n is even, so $n = 2k$ for some $n \in \mathbb{Z}$. But then $n^2 = (2k)^2 = 2 \cdot (2k^2)$, so n^2 is even, and therefore not odd. Therefore, n^2 odd $\Rightarrow n$ is odd. $\square$

5. Use proof by contrapositive to prove the following.

Proposition. Let $a, b \in \mathbb{Z}$. If ab is a multiple of 3, then either a or b is a multiple of 3. (Hint: an integer that is *not* a multiple of 3 is of the form $3k + r$ where $k \in \mathbb{Z}$ and r equals either 1 or 2.)

Proof. Let $a, b \in \mathbb{Z}$, and suppose neither a nor b is a multiple of 3. Then, by the hint, $a = 3k + r$ and $b = 3\ell + s$, where $k, \ell \in \mathbb{Z}$, r equals 1 or 2, and s equals 1 or 2. But then

$$ab = (3k + r)(3\ell + s) = 9k\ell + 3ks + 3\ell r + rs = 3 \cdot (3k\ell + ks + \ell r) + rs.$$

Now note that, since r and s are each equal to 1 or 2, rs equals 1, 2, or 4. If $rs = 1$, then $ab = 3 \cdot (3k\ell + ks + \ell r) + 1$, and therefore ab is not a multiple of 3, by the hint. If $rs = 2$, then $ab = 3 \cdot (3k\ell + ks + \ell r) + 2$, and therefore ab is not a multiple of 3, again by the hint. If $rs = 4$, then

$$ab = 3 \cdot (3k\ell + ks + \ell r) + 4 = 3 \cdot (3k\ell + ks + \ell r + 1) + 1,$$

so ab is not a multiple of 3, by the hint.

Therefore, if ab is a multiple of 3, then either a or b is a multiple of 3. $\square$

6. Prove that, if $m \in \mathbb{Z}$ is even, then $4|m^2$, and if m is odd, then $4|(m^2 - 1)$.

Proof. Suppose $m \in \mathbb{Z}$ is even. Then $m = 2k$ for some $k \in \mathbb{Z}$, so $m^2 = (2k)^2 = 4k^2$, so $4|m^2$. Now suppose $m \in \mathbb{Z}$ is odd. Then $m = 2k + 1$ for some $k \in \mathbb{Z}$, so

$$m^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k),$$

so $4|(m^2 - 1)$.

Therefore, if $m \in \mathbb{Z}$ is even, then $4|m^2$, and if $m \in \mathbb{Z}$ is odd, then $4|(m^2 - 1)$. $\square$

7. Prove by contradiction that, if $a, b \in \mathbb{Z}$ are odd, then there does not exist $c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$. Hint: assume $a, b \in \mathbb{Z}$ are odd and that there does exist such an integer c. Show that $4|(c^2-2)$. Use this together with the previous exercise to derive a contradiction.

Proof. Assume $a, b \in \mathbb{Z}$ are odd, and $\exists c \in \mathbb{Z} : a^2 + b^2 = c^2$. Since a and b are odd, $\exists k, \ell \in \mathbb{Z}$ with $a = 2k + 1$ and $b = 2\ell + 1$. Then

$$c^2 - 2 = a^2 + b^2 - 2 = (2k+1)^2 + (2\ell+1)^2 - 2 = 4k^2 + 4k + 1 + 4\ell^2 + 4\ell + 1 - 2 = 4(k^2 + k + \ell^2 + \ell),$$

so $4|(c^2-2)$. Now if c^2 is even, then $4|c^2$ by the previous exercise, so by Exercise 1(a) above, $4|(c^2 - (c^2 - 2))$; that is, $4|2$, a contradiction (since $4 \nmid 2$). On the other hand, if c^2 is odd, then $4|(c^2 - 1)$ by the previous exercise, so by Exercise 1(a) above, $4|((c^2 - 1) - (c^2 - 2))$; that is, $4|1$, a contradiction (since $4 \nmid 1$).

So if $a, b \in \mathbb{Z}$ are odd, then there does not exist $c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$. $\square$

8. Identify each of the following statements as true or false (circle "**T**" or "**F**"). **Please explain your answers:** If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement, and explain why it's a counterexample.

(a) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon (m-n)|k$. **T** **F**

Given m and n, let $k = m - n$.

(b) $\exists k \in \mathbb{Z}\colon \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m-n)|k$. **T** **F**

(Let $k = 0$.)

(c) $\sim\big(\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon \forall n \in \mathbb{Z}, (m-n)|k\big)$. **T** **F**

Consider the statement $\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon \forall n \in \mathbb{Z}, (m-n)|k$. This statement is true because, given m, let $k = 0$. Then for any n, $(m-n)|k$ since, again, everything divides 0. So the negation of this statement is false.

9. Let F_n be the nth Fibonacci number, defined by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \quad (n \geq 1).$$

Use mathematical induction to prove that

$$F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1.$$

Proof. Let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true?

$$F_1 \stackrel{?}{=} F_3 - 1$$
$$1 = 2 - 1 = 1,$$

so $A(1)$ is true.

Step 2: Assume

$$A(k):\ F_1 + F_2 + F_3 + F_4 + \cdots + F_k = F_{k+2} - 1.$$

Then

$$\begin{aligned} &F_1 + F_2 + F_3 + F_4 + \cdots + F_{k+1} \\ &= (F_1 + F_2 + F_3 + F_4 + \cdots + F_k) + F_{k+1} \\ &= F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1 \end{aligned}$$

(at the last step, we used the fact that, by the recursive formula for Fibonacci numbers, $F_{k+2} + F_{k+1} = F_{k+3}$), so $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all n.

10. Use the principle of mathematical induction to prove that, for any $n \in \mathbb{N}$,

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

(Hint: $(k+2)(k+1)! = (k+2)!$.) Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof.

Proof. Let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1 \cdot 1! &\overset{?}{=} (1+1)! - 1 \\ 1 &= 1, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \ 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1.$$

Then

$$\begin{aligned} &1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (k+1) \cdot (k+1)! \\ &= \big(1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k!\big) + (k+1) \cdot (k+1)! \\ &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (1 + k + 1)(k+1)! - 1 \\ &= (k+2)(k+1)! - 1 = (k+2)! - 1, \end{aligned}$$

so $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all n.

11. Use the principle of mathematical induction to prove that, for any $n \in \mathbb{N}$,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Proof. Let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1 \cdot 3 &\overset{?}{=} \frac{1(1+1)(2 \cdot 1 + 7)}{6} \\ 3 &= \frac{1 \cdot 2 \cdot 9}{6} = 3, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \ 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}.$$

Then

$$\begin{aligned}
&1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+(k+1)(k+3)\\
&=\big(1\cdot 3+2\cdot 4+3\cdot 5+4\cdot 6+\cdots+k(k+2)\big)+(k+1)(k+3)\\
&=\frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)\\
&=\frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6}=\frac{(k+1)\big(k(2k+7)+6(k+3)\big)}{6}\\
&=\frac{(k+1)(2k^2+13k+18)}{6}=\frac{(k+1)(k+2)(2k+9)}{6}\\
&=\frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+7\big)}{6},
\end{aligned}$$

so $A(k+1)$ follows. So $A(k)\Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all n.

12. Prove that, given any natural number $n\in\mathbb{N}$ with $n\geq 8$, there exist integers $a,b\in\mathbb{Z}_{\geq 0}=\{0,1,2,3,\ldots,\}$ such that

$$3a+5b=n.$$

(In other words, prove that any postage amount of 8 cents or more can be made from 3 cent and 5 cent stamps only.) Use the following version of strong induction: Let $A(n)$ be the given statement.

- Prove that $A(8)$, $A(9)$, and $A(10)$ are true.
- Prove that $A(k)\Rightarrow A(k+3)$ for $k\geq 8$.

Explain (at least intuitively) why this is enough.

Proof. Let $A(n)$ be the statement in question.

Step 1: Are $A(8), A(9), A(10)$ true? Yes:

$$3\cdot 1+5\cdot 1=8;\qquad 3\cdot 3+5\cdot 0=9,\qquad 3\cdot 0+5\cdot 2=10.$$

Step 2: Assume

$$A(k):\ \exists a,b\in\mathbb{Z}_{\geq 0}\ni 3a+5b=k.$$

Then

$$3(a+1)+5b=3a+5b+3=k+3,$$

so $A(k+3)$ follows. So $A(k)\Rightarrow A(k+3)$.

(The idea is: Since $A(8)$ is true and $A(k)\Rightarrow A(k+3)$, we know that $A(11), A(14), A(17), A(20),\ldots$ are true. Since $A(9)$ is true and $A(k)\Rightarrow A(k+3)$, we know that $A(12), A(15), A(18), A(21),\ldots$ are true. Since $A(10)$ is true and $A(k)\Rightarrow A(k+3)$, we know that $A(13), A(16), A(19), A(22),\ldots$ are true. So "ultimately," we get everything.) So by the principle of (strong) mathematical induction, $A(n)$ is true $\forall n\in\mathbb{N}$ with $n\geq 8$. □

13. Identify each of the following statements as true or false, by putting a "T" or "F" in the space to the *left* of the statement. Then, in the space to the *right* of the statement, put the *number* of the statement that is the *negation* of the statement in question. For example, if the negation of statement 2 is statement 7, then put a "7" in the space to the right of statement 2.

One of the statements has no negation present, so leave the space to the right of that statement blank.

(Recall that $\mathbb{R}^+$ denotes the set of positive real numbers.)

1.	F	$\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$	3
2.	F	$\exists w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$	7
3.	T	$\exists w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}^+, x < w + y + z$	1
4.	T	$\sim(\sim(\forall w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}^+, x - y < w + z))$	5
5.	F	$\exists w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$	4
6.	F	$\forall w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}^+, w + z \leq x - y$	
7.	T	$\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \sim(\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y)$	2
8.	F	$\sim(\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \sim(\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, x - y < w + z))$	9
9.	T	$\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}^+, w + z \leq x - y$	8

14. Let $f(x) = 3x - 7$.

(a) Prove that, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - 4| < \delta \Rightarrow |f(x) - 5| < \varepsilon.$$

(b) Restate what you just proved in terms of a limit of $f(x)$.

Proof. Let $\varepsilon > 0$. [Scratchwork: We want

$$|f(x) - 5| = |3x - 7 - 5| = |3x - 12| = 3|x - 4| < \varepsilon.$$

So we want $|x - 4| < \varepsilon/3$. So write this.] Let $\delta = \varepsilon/3$. Then

$$|x - 4| < \delta \Rightarrow |f(x) - 5| = |3x - 7 - 5| = |3x - 12| = 3|x - 4| < 3 \cdot (\varepsilon/3) = \varepsilon.$$

Therefore, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - 4| < \delta \Rightarrow |f(x) - 5| < \varepsilon. \qquad \square$$

(b)

$$\lim_{x\to 4} f(x) = 5.$$

15. Let $g(x) = x^2 - 1$.

(a) Prove that, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$|x-1| < \delta \Rightarrow |g(x)| < \varepsilon.$$

Hint: let $\delta = \min\{\varepsilon/3, 1\}$ (the minimum of $\varepsilon/3$ and 1).

(b) Restate what you just proved in terms of a limit of $g(x)$.

Proof. Let $\varepsilon > 0$. [Scratchwork: We want

$$|g(x)| = |x^2 - 1| = |(x-1)(x+1)| = |x-1| \cdot |x+1|.$$

Suppose, as in the hint, we let $\delta = \min\{\varepsilon/3, 1\}$. Then if $|x-1| < \delta$, we certainly have $|x-1| < \varepsilon/3$ as well as $|x-1| < 1$. But note that $|x-1| < 1$ is the same as $-1 < x-1 < 1$, which gives $1 < x+1 < 3$, which certainly implies $|x+1| < 3$. So write this.] Let $\delta = \min\{\varepsilon/3, 1\}$. Then

$$|x-1| < \delta \Rightarrow |g(x)| = |(x-1)(x+1)| = |x-1| \cdot |x+1| < (\varepsilon/3) \cdot 3 = \varepsilon.$$

Therefore, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that

$$|x-1| < \delta \Rightarrow |g(x)| < \varepsilon.$$

(b)

$$\lim_{x\to 1} g(x) = 0.$$

16. (a) Explain, intuitively, why the negation of the statement $P \Rightarrow Q$ is the statement $P \wedge \sim Q$ (meaning "P and not Q"). The statement $P \Rightarrow Q$ means whenever P happens, Q must happen too. So the negation of this would be the case where P happens but Q does not. This is $P \wedge \sim Q$.

(b) Negate the statement

$$\forall\varepsilon > 0, \exists\delta > 0 \ni |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

$$\exists\varepsilon > 0 : \forall\delta > 0, |x-a| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

17. Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.

Proof. Let $A(n)$ be the statement $5^{2n} - 1 = 8m$ for some $m \in \mathbb{Z}$.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 5^{2\cdot 1} - 1 &\stackrel{?}{=} 8m \\ 24 &= 8 \cdot 3, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : \; 5^{2k} - 1 = 8m$$

for some $m \in \mathbb{Z}$. Then

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^2 \cdot 5^{2k} - 1 \\ &= 5^2 \cdot (5^{2k} - 1) + 5^2 - 1 \\ &= 5^2 \cdot (5^{2k} - 1) + 24 \\ &= 5^2 \cdot (8m) + 8 \cdot 3 = 8(25m + 3), \end{aligned}$$

so $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all n.

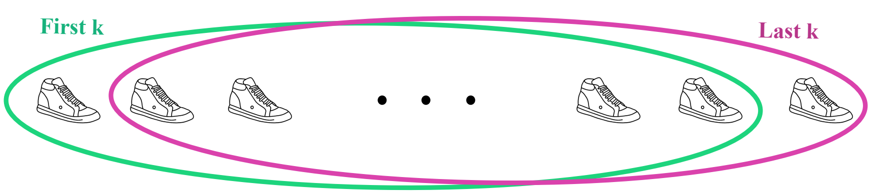
18. What's wrong with the following proof? You tell me.

Theorem. All sneakers are identical.

Proof. Let $A(n)$ be the statement that all sneakers in any set of n sneakers are identical.

Is $A(1)$ true? Yes, any one sneaker is identical to itself.

Now assume $A(k)$: any k sneakers are identical to each other. To deduce $A(k+1)$, line all $k+1$ sneakers up in a row. The first k sneakers in that row are identical, by the induction hypothesis. So are the last k, by the induction hypothesis. But the second sneaker in the row belongs to both the first k and the last k, so all sneakers are identical to the second one.



So $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$. So by induction, $A(n)$ is true for all $n \in \mathbb{N}$.

In particular, let n be the total number of sneakers in existence. Then all of these are identical. $\square$