1. Quantifiers.

(a) The quantifier "$\forall$" means "for all," or "for each," or "for every."
If X is a set and $Q(x)$ is a statement about a quantity x, then the statement

$$\forall x \in X : Q(x)$$

means the statement $Q(x)$ is true for every x in X.

(b) The quantifier "$\exists$" means "for some," or "for at least one," or "there exists."
If X is a set and $Q(x)$ is a statement about a quantity x, then the statement

$$\exists x \in X : Q(x)$$

means the statement $Q(x)$ is true some (at least one, possibly more) x in X.

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2. Proof templates.

(a) $P \Rightarrow Q$, direct proof.

Theorem. $P \Rightarrow Q$.
Proof. Assume P. [Now do what you need to conclude:] Therefore, Q.
So $P \Rightarrow Q$. $\square$

(b) $P \Rightarrow Q$, contrapositive proof.

Theorem. $P \Rightarrow Q$.
Proof. Assume $\sim Q$. [Now do what you need to conclude:] Therefore, $\sim P$.
So $P \Rightarrow Q$. $\square$

(c) $P \Leftrightarrow Q$.

Theorem. $P \Leftrightarrow Q$.
Proof. Assume P. [Now do what you need to conclude:] Therefore, Q.
So $P \Rightarrow Q$.
Next, assume Q. [Now do what you need to conclude:] Therefore, P.
So $Q \Rightarrow P$.
Therefore, $P \Leftrightarrow Q$. $\square$

(d) Proofs with universal quantifiers.

Theorem. $\forall x \in X$, $Q(x)$.
Proof. Assume $x \in X$. [Now do what you need to conclude:] Therefore, $Q(x)$.
So $\forall x \in X$, $Q(x)$. $\square$

(e) Proofs with existential quantifiers.

Theorem. $\exists x \in X$, $Q(x)$.
Proof. [Find a particular $x \in X$, call it x_0, that has the property $Q(x)$. Then write:] Let $x = x_0$. Then …[show that $Q(x_0)$ is true]. So $\exists x \in X$, $Q(x)$. $\square$

(f) Proof by contradiction.

Theorem. T.
Proof. Assume $\sim T$. [Then do what's necessary to derive a contradiction, and write:] Contradiction. Therefore T is true. $\square$

(g) Proof by the principle of mathematical induction.

Theorem. $\forall n \in \mathbb{N},\ A(n)$.
Proof. Step 1: Is $A(1)$ true? [Now do what you need to conclude:] So $A(1)$ is true.
Step 2: Assume $A(k)$. [Now do what you need to conclude:] So $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.
Therefore, by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

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3. Some special sets.

(a) $\mathbb{Z} = \{\text{integers}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

(b) $\mathbb{N} = \{\text{natural numbers}\} = \{1, 2, 3, \ldots\}$.

(c) $\mathbb{R} = \{\text{real numbers}\} = (-\infty, \infty)$.

(d) $\mathbb{Q} = \{\text{rational numbers}\} = \{\text{fractions } m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

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4. Facts about integers.

(a) Let $a, b \in \mathbb{Z}$. We say a divides b, written $a|b$, if $b = na$ for some $n \in \mathbb{Z}$.

(b) (Division algorithm.) Given integers a and b with $b > 0$, there exist unique integers q and r for which $a = qb + r$ and $0 \leq r < b$.