MATH 3001-001: Analysis I

September 17, 2025

First In-class Midterm Exam: SOLUTIONS

1. (20 points) Prove the following:

Proposition. Let $a, b, c, d, m \in \mathbb{Z}$. If $m|(a-b)$ and $m|(c-d)$, then $m|(ac-bd)$.

(Use the definition of "|" ("divides") from item 4(a) of your fact sheet. You can also use the fact that sums, differences, and products of integers are integers. But don't use any other facts about divisibility.)

Hint: $ac - bd = c(a-b) + b(c-d)$.

Proof.

Assume that $a, b, c, d, m \in \mathbb{Z}$, and that $m|(a-b)$ and $m|(c-d)$. By definition of divisibility, we have $a - b = mk$ and $c - d = m\ell$ for some $k, \ell \in \mathbb{Z}$. But then, by the hint,

$$ac - bd = c(a-b) + b(c-d) = c \cdot mk + b \cdot m\ell = m(ck + b\ell).$$

Since c, k, b, and ℓ are integers, so is $ck + b\ell$. So by definition of divisibility, $m|(ac - bd)$.

So $a, b, c, d, m \in \mathbb{Z}$, $m|(a-b)$, and $m|(c-d) \Rightarrow m|(ac-bd)$. □

2. (20 points; 5 points each) Form the negation of each of the following statements. Your negated statement should not contain the symbol "∼" (or any symbol that means "not" or "negation") in it anywhere. (Don't worry about whether these statements are true or false.) Remark: the negation of $a < b$ is $a \geq b$.

(a) $\forall z \in \mathbb{R}, |25 - 37| < z$.

$\exists z \in \mathbb{R} \ni |25 - 37| \geq z$.

(b) $\exists y \in \mathbb{R} \ni \forall z \in \mathbb{R}, |25 - y| < z$.

$\forall y \in \mathbb{R}, \exists z \in \mathbb{R} \ni |25 - y| \geq z$.

(c) $\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \ni \forall z \in \mathbb{R}\colon |x - y| < z)$.

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \ni \forall z \in \mathbb{R}, |x - y| < z$.

(d) $\forall x \in \mathbb{R}, \sim(\exists y \in \mathbb{R} : \sim(\forall z \in \mathbb{R}\colon |x - y| < z))$.

$\exists x \in \mathbb{R} \ni \exists y \in \mathbb{R} \ni \exists z \in \mathbb{R} \ni |x - y| \geq z$.

3. (20 points; 5 points each) Identify each of the following statements as true or false (circle "**T**" or "**F**"). **Please explain your answers:** If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement, and explain why it's a counterexample.

(a) $\exists m \in \mathbb{Z} : \exists n \in \mathbb{Z} : \exists k \in \mathbb{Z} : \ (m-n)|k.$ **T** **F**

For example, let $m = 4$, $n = -7$, and $k = 22$: then $m - n = 11$, and $11|22$, so $(m-n)|k$.

(b) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} : \ (m-n)|k.$ **T** **F**

Given m and n, let $k = m - n$. Every number divides itself, so $(m-n)|k$. (Or let $k = 0$.)

(c) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} : k|(m-n).$ **T** **F**

As in the previous part of this exercise, let $k = m - n$. Then $(m-n)|k$.

(d) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z}, k|(m-n).$ **T** **F**

The statement fails for $m = 2$, $n = 1$, and $k = 37$.

4. (20 points) Use the principle of mathematical induction to prove the following. Please supply a complete proof; a series of unconnected calculations will not suffice. For example, if you make a reference to a statement $A(n)$, please state what $A(n)$ *is*, though it's OK to say something like "let $A(n)$ be the statement in question." Also make sure you clearly state your conclusion, and so on.

Theorem. $\forall n \in \mathbb{N}$,

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

Proof. Let $A(n)$ be the statement in question.

Step 1. Is $A(1)$ true?

$$1 \stackrel{?}{=} 1^2$$

$$1 = 1,$$

so $A(1)$ is true.

Step 2. Assume

$$A(k) : 1 + 3 + 5 + \cdots + 2k - 1 = k^2.$$

Then

$$\begin{aligned} 1 + 3 + 5 + \cdots + 2(k+1) - 1 &= 1 + 3 + 5 + \cdots + 2(k+1) - 1 \\ &= (1 + 3 + 5 + \cdots + 2k - 1) + 2(k+1) - 1 \\ &= k^2 + 2k + 1 = (k+1)^2, \end{aligned}$$

so $A(k+1)$ follows.

So $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction,

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

$\forall n \in \mathbb{N}$. □

5. (20 points; 1 point for each of the 20 blanks) Let

$$f(x) = x^2 + x - 7.$$

(a) Fill in the blanks to complete the following proof. Hint: $x^2 + x - 12 = (x-3)(x+4)$.

Proposition. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - 3| < \delta \Rightarrow |f(x) - 5| < \varepsilon.$$

Proof. Let $\underline{\quad \varepsilon \quad} > 0$. Let $\delta = \min\{1, \varepsilon/8\}$ (the minimum of 1 and $\varepsilon/8$). Note that, if $|x-3| < \delta$, then $|x - 3| < 1$ (since $\delta \leq \underline{\quad 1 \quad}$), so $-1 < x - 3 < \underline{\quad 1 \quad}$, so, adding 7 to all parts of this inequality, $6 < x + 4 < \underline{\quad 8 \quad}$, which certainly tells us that $|x + 4| < 8$. So

$$|x-3| < \delta \Rightarrow |f(x) - 5| = |x^2 + x - 7 - 5| = |x^2 + x - 12| = |(x - \underline{\quad 3 \quad})(x+4)|$$

$$= |x - 3| \cdot |\underline{\quad x + 4 \quad}| < \delta \cdot 8 \leq (\underline{\quad \varepsilon \quad}/8) \cdot 8 = \underline{\quad \varepsilon \quad}.$$

Therefore, $\forall \varepsilon > 0$, $\exists \delta > \underline{\quad 0 \quad}$ such that

$$|x - 3| < \delta \Rightarrow \underline{\quad |f(x) - 5| \quad} < \varepsilon.$$

ATWMR

(For the last blank, fill in your own end-of-proof tag.)

(over)

(b) Fill in the blanks: Let's summarize what we've shown on the previous page. We've shown that we can assure that $|f(x) - 5| < \varepsilon$, for any positive number ε, as long as x is within $\delta = \min\{\underline{\quad 1 \quad}, \varepsilon/8\}$ units of 3. Now think of ε as being a *small* positive number. Then what we've shown is: we can make $f(x)$ as close as we want to 5 — specifically, we can assure that $f(x)$ is within $\underline{\quad \varepsilon \quad}$ units of 5 — as long as we choose x close enough to $\underline{\quad 3 \quad}$ — that is, within δ units of $\underline{\quad 3 \quad}$.

So: no matter how close we want $f(x)$ to be to 5, we can achieve that, if we choose x close enough to $\underline{\quad 3 \quad}$. Or in other words: x being sufficiently close to 3 guarantees that $f(x)$ is as close as desired to $\underline{\quad 5 \quad}$. So what we've just proved is that

$$\lim_{x \to \underline{\quad 3 \quad}} \underline{\quad f(x) \quad} = \underline{\quad 5 \quad}.$$