Sequences and limits. (Sec. 4.1.)

(I) Notation.

By (s_n), we mean the sequence

$$s_1, s_2, s_3, \ldots$$

Technically, a sequence (s_n) is a function $s: \mathbb{N} \to \mathbb{R}$, where we write s_n for $s(n)$.

More generally, $(s_n)_{n=m}^{\infty}$ denotes the sequence

$s_m, s_{m+1}, s_{m+2}, \ldots$. E.g. $\left(\frac{1}{1+n}\right)_{n=6}^{\infty}$ denotes

$$\frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \ldots$$

(which could also be denoted $\left(\frac{1}{6+n}\right)$).

(II) Limits of sequences.

We say

$$\lim_{n\to\infty} s_n = L$$

if, $\forall\, \varepsilon > 0$, $\exists N \in \mathbb{N}: n \geq N \Rightarrow |s_n - L| < \varepsilon$.

Example 1.

Show that $\lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1$.

Solution.

Let $\varepsilon > 0$. [We want

$$\left| 1 + \frac{(-1)^n}{n} - 1 \right| = \left| {(-1)^n}/{n} \right| = \frac{1}{n} < \varepsilon.$$

This will happen if $n > 1/\varepsilon$. So:]

Let $N \in \mathbb{N}$ be any integer $> \varepsilon$. If $n \geq N$, then

$$\left| 1 + \frac{(-1)^n}{n} - 1 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So

$$\lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1. \qquad \square$$

We have:

<u>Theorem 4.1.8.</u> If $|s_n - L|$ is bounded by a sequence that converges to zero, then s_n converges to L.

More formally:

If there is a constant $C > 0$ (independent of n) and an integer $m \in \mathbb{N}$ such that

$$|s_n - L| \leq C|a_n|$$

for all $n \geq m$, and if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} s_n = L$.

<u>Proof.</u> Assume $|s_n - L| < \varepsilon \; \forall n \geq m$, and $\lim_{n\to\infty} a_n = 0$.

Let $\varepsilon > 0$. We know $\exists N \in \mathbb{N}: n \geq N \Rightarrow |a_n| < \varepsilon / C$. But then, for $n \geq \max\{m, N\}$,

$$|s_n - L| \leq C|a_n| < C \cdot \varepsilon / C = \varepsilon,$$

so $\lim_{n\to\infty} s_n = L$. $\square$

<u>Example 2.</u>

Show that $\displaystyle\lim_{n\to\infty} \frac{7n^3 + 24n}{10n^3 + 31n^2} = \frac{7}{10}$.

<u>Proof</u>

Note that $\displaystyle\left| \frac{7n^3 + 24n^2}{10n^3 + 31n^2} - \frac{7}{10} \right| = \frac{23n^2}{10(10n^3 + 31n^2)}$.

Since $10n^3+31n^2 > 10n^3$, we have

$$\left|\frac{7n^3+24n^2}{10n^3+31n^2} - \frac{7}{10}\right| \leq \frac{23n^2}{100n^3} = \frac{23}{100n}.$$

Let $\varepsilon > 0$. If $N > \frac{23}{100\varepsilon}$, then $n \geq N \Rightarrow$

$$\left|\frac{7n^3+24n^2}{10n^3+31n^2} - \frac{7}{10}\right| \leq \frac{23}{100n} \leq \frac{23}{100N} < \frac{23}{100(23/(100\varepsilon))} = \varepsilon.$$

So $\lim\limits_{n\to\infty} \frac{7n^3+24n^2}{10n^3+31n^2} = \frac{7}{10}$. $\square$