Compact sets (Sec. 3.5).

Definition 3.5.1.
We say $S \subseteq \mathbb{R}$ is compact if, for every collection $\{T_\alpha : \alpha \in A\}$ of open sets $T_\alpha \subseteq \mathbb{R}$ (A can be any indexing set) such that

$$S \subseteq \bigcup_{\alpha \in A} T_\alpha,$$

there is a finite set $B \subseteq A$ such that

$$S \subseteq \bigcup_{\alpha \in B} T_\alpha.$$

In this context, we call $\{T_\alpha : \alpha \in A\}$ an open cover of S, and $\{T_\alpha : \alpha \in B\}$ a finite subcover.

So, "S is compact" means: every open cover of S has a finite subcover.

Examples.
(i) $(0,1)$ is not compact. Proof: We have

$$(0,1) = \bigcup_{n=1}^{\infty} (1/n, 1).$$

But consider any collection

$$\{(1/n, 1) : n \in B\},$$

where B is a finite set of positive integers. Let $m = \max B$.* Then

$$\bigcup_{n \in B} (1/n, 1) = (1/m, 1),$$

But $(0,1) \not\subseteq (1/m, 1)$. So the open cover $\{(1/n, 1) : n \in \mathbb{N}\}$ of $(0,1)$ has no finite subcover. So $(0,1)$ is not compact.

* Every finite subset of $\mathbb{N}$ has a maximum. (Why?)

(ii) $[0,1]$ is compact. Proof: it follows from

<u>Thm. 3.5.5</u> (Heine-Borel Theorem.)
$S \subseteq \mathbb{R}$ is compact iff S is closed and bounded.

<u>Proof</u> that compact $\Rightarrow$ closed and bounded (for the converse, see text).

Assume $S \subseteq \mathbb{R}$ is compact. Since the collection $\{(-n, n) : n \in \mathbb{N}\}$ covers $\mathbb{R}$, it also covers S. Since S is compact, some subcollection

$$\{(-n, n) : n \in B\},$$

where B is finite, covers S. Let $m = \max B$.

Then $(-n, n) \subseteq (-m, m)$ $\forall n \in B$, so

$$S \subseteq \bigcup_{n \in B} (-n, n) = (-m, m).$$

So S is bounded below by $-m$ and above by m.

To show S is closed, it suffices to show that $\mathrm{bd}\, S \subseteq S$. Suppose not: then $\exists x \in \mathrm{bd}\, S$ with $x \notin S$.

Note that $\{x\} = \bigcap_{n=1}^{\infty} V_n$, where $V_n = [x - 1/n, x + 1/n]$.

Define $U_n = \mathbb{R} \setminus V_n$: since each V_n is closed, each U_n is open. Also, since $x \notin S$, we have

$$S \subseteq \mathbb{R} \setminus \{x\} = \mathbb{R} \setminus \left(\bigcap_{n=1}^{\infty} V_n \right) = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus V_n)$$
$$= \bigcup_{n=1}^{\infty} U_n.$$

So $\{ U_n : n \in \mathbb{N} \}$ is an open cover of S: since S is compact, $\exists$ a finite subset B of $\mathbb{N}$ with

$$S \subseteq \bigcup_{n \in B} U_n.$$

But then, taking complements,

$$\mathbb{R} \setminus S \supseteq \bigcap_{n \in B} (\mathbb{R} \setminus U_n) = \bigcap_{n \in B} V_n$$
$$= \bigcap_{n \in B} [x - 1/n, x + 1/n].$$

The intersection on the right equals $[x - 1/m, x + 1/m]$, where $m = \max B$.

But $N(x, 1/m) \subseteq [x - 1/m, x + 1/m]$. So $N(x, 1/m)$ lies outside of S, contradicting the fact that $x \in \operatorname{bd} S$.

So $\operatorname{bd} S \subseteq S$, so S is closed.

$\square$

A consequence:

Theorem 3.5.6.

(Bolzano - Weierstrass). $S \subseteq \mathbb{R}$ has infinitely many elements $\Rightarrow$ either S is unbounded, or S has at least one accumulation point in $\mathbb{R}$.

Proof omitted.