I) Open and closed sets (sec. 3.4), continued:

Accumulation points.

Some definitions:

1) Let $\varepsilon > 0$ and $x \in \mathbb{R}$. We define the deleted nbhd $N^*(x,\varepsilon)$ by

$$N^*(x,\varepsilon) = N(x,\varepsilon) \setminus \{x\} = (x-\varepsilon, x) \cup (x, x+\varepsilon).$$

2) We say $x \in \mathbb{R}$ is an accumulation point for S if every deleted nbhd of x intersects S – that is, if

$$\forall \varepsilon > 0,\ N^*(x,\varepsilon) \cap S \neq \emptyset.$$

The set of all accumulation points of S is denoted S'.

Remark: We may think of an accumulation point of S as "a point (in $\mathbb{R}$) at which S bunches up."

3) If $x \in S$ but $x \notin S'$ (i.e. $x \in S \setminus S'$), we say x is an isolated point of S.

4) We define the closure of S, denoted $\mathrm{cl}\, S$, by

$$\mathrm{cl}\, S = S \cup S'.$$

Example

Find the interior, boundary, accumulation and isolated points, and closure, of:

(i) $U = (-2,3) \cup \{4\} \cup (5,\infty)$ (ii) $V = \bigcup_{n=1}^{\infty} (n, n+1)$

(iii) $X = \bigcap_{n=1}^{\infty} (-1/n, 1/n)$ (iv) $Y = \{1/n : n \in \mathbb{N}\}$

<u>Solution.</u>

(i) $\text{int}\, U = (-2,3) \cup (5,\infty)$, $\text{bd}\, U = \{-2,3,4,5\}$,
$U' = [-2,3] \cup [5,\infty)$, $U \setminus U' = \{4\}$,
$\text{cl}\, U = [-2,3] \cup \{4\} \cup [5,\infty)$

(ii) $\text{int}\, V = V$, $\text{bd}\, V = \mathbb{N}$, $V' = [1,\infty)$, $V \setminus V' = \emptyset$,
$\text{cl}\, V = V'$

(iii) Note that $X = \{0\}$. So $\text{int}\, X = \emptyset$, $\text{bd}\, X = X$,
$X' = \emptyset$, $X \setminus X' = \{0\}$, $\text{cl}\, X = X$.

(iv) $\text{int}\, Y = \emptyset$, $\text{bd}\, Y = Y \cup \{0\}$, $Y' = \{0\}$, $Y \setminus Y' = Y$,
$\text{cl}\, Y = Y \cup \{0\}$.

Note that, in general, $S' \not\subseteq \text{bd}\, S$ and $\text{bd}\, S \not\subseteq S'$.

But we do have:

<u>Theorem 3.4.17.</u> Let $S \subseteq \mathbb{R}$.

(a) S is closed $\Leftrightarrow$ $S' \subseteq S$.

(b) $\text{cl}\, S$ is closed.

(c) S is closed $\Leftrightarrow$ $S = \text{cl}\, S$.

(d) $\text{cl}\, S = S \cup \text{bd}\, S$.

<u>Proof</u> (of some parts). Let $S \subseteq \mathbb{R}$.

To prove the "$\Rightarrow$" direction of (a): assume S is closed. We show that $\mathbb{R} \setminus S \subseteq \mathbb{R} \setminus S'$ as follows.

Let $x \in \mathbb{R} \setminus S$. Since $\mathbb{R} \setminus S$ is open, $\exists \varepsilon > 0$: $N(x,\varepsilon) \subseteq \mathbb{R} \setminus S$. But $N^*(x,\varepsilon) \subseteq N(x,\varepsilon)$, so $N^*(x,\varepsilon) \subseteq \mathbb{R} \setminus S$. So $N^*(x,\varepsilon)$ does not intersect S, so x is not an accumulation point of S.

To prove the "$\Rightarrow$" direction of (c): suppose S is closed. Then by part (a), $S' \subseteq S$, which implies $S' \cup S = S$. But $S' \cup S = \mathrm{cl}\, S$ by definition, so $\mathrm{cl}\, S = S$.

For part (d), we show $\mathrm{cl}\, S \subseteq S \cup \mathrm{bd}\, S$; the proof that $S \cup \mathrm{bd}\, S \subseteq \mathrm{cl}\, S$ is similar.

So let $x \in \mathrm{cl}\, S$. Then by definition, $x \in S$ or $x \in S'$. If $x \in S$ then certainly $x \in S \cup \mathrm{bd}\, S$, so we're done. So assume $x \notin S$: since $x \in \mathrm{cl}\, S$, we must have $x \in S'$.

Let $\varepsilon > 0$. Then $N^*(x,\varepsilon)$ intersects S in at least one point, call it y. But then $N(x,\varepsilon)$ intersects S at y and $\mathbb{R} \setminus S$ at x. So $x \in \mathrm{bd}\, S$. So $x \in S \cup \mathrm{bd}\, S$.

So $\mathrm{cl}\, S \subseteq S \cup \mathrm{bd}\, S$.

$\square$