The topology of $\mathbb{R}$.

"study of open and closed sets"

Definitions 3.4.1 - 3.4.3 and 3.4.6.

A) Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Then:

The neighborhood (nbhd) $N(x,\varepsilon)$ of x is defined by

$$N(x,\varepsilon) = \{y \in \mathbb{R} : |x-y| < \varepsilon\} = (x-\varepsilon, x+\varepsilon).$$

B) Let $S \subseteq \mathbb{R}$.

1) We say $x \in \mathbb{R}$ is an interior point of S if $\exists$ a nbhd $N(x,\varepsilon)$ with $N(x,\varepsilon) \subseteq S$.

2) If every nbhd N of $x \in \mathbb{R}$ intersects both S and $\mathbb{R} \setminus S$, then x is a boundary point of S.

3) We write $\operatorname{int} S$ for the set of interior points of S, and $\operatorname{bd} S$ for the set of boundary points of S.

C) If $\operatorname{bd} S \subseteq S$, we say S is closed. If $\operatorname{bd} S \subseteq \mathbb{R} \setminus S$, we say S is open.

Examples:

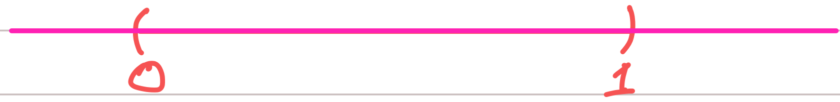
interior points (each has a nbhd $N \subseteq S$)

a) $S = [0,1]$

boundary points (any nbhd intersects both S and $\mathbb{R} \setminus S$)

$[0,1]$ is closed, as $\operatorname{bd} S = \{0,1\} \subseteq S$

b) $T=(0,1)$



$\operatorname{bd} T = \{0,1\} \subseteq \mathbb{R}\setminus T$, so T is open.

c) $U=[0,1)$ is neither open nor closed.

Similarly, $[a,b]$ is closed, (a,b) open, and $[a,b)$ and $(a,b]$ neither, for $-\infty < a \le b < \infty$. $(-\infty,a)$ and (b,∞) are open; $(-\infty,a]$ and $[b,\infty)$ are closed. $\emptyset$ and $\mathbb{R}$ are open <u>and</u> closed (why)?

Note that, in general,

(i) $\operatorname{int} S \subseteq S$;

(ii) every $x \in S$ is either in $\operatorname{int} S$ or $\operatorname{bd} S$ (but not both);

(iii) $\operatorname{bd} S = \operatorname{bd}(\mathbb{R}\setminus S)$.

<u>Theorem 3.4.7.</u>

(a) The set S is open iff $S = \operatorname{int} S$.

(b) The set S is closed iff $\mathbb{R}\setminus S$ is open.

<u>Proof.</u>

(a) S is open $\Leftrightarrow$ $\operatorname{bd} S \subseteq \mathbb{R}\setminus S$ $\Leftrightarrow$ (by note (ii) above) every $x \in S$ is in $\operatorname{int} S$ $\Leftrightarrow$ $S \subseteq \operatorname{int} S$ $\Leftrightarrow$ (by note (i) above) $S = \operatorname{int} S$.

(b) S is closed $\Leftrightarrow$ $\operatorname{bd} S \subseteq S$ $\Leftrightarrow$ (by note (iii) above) $\operatorname{bd}(\mathbb{R}\setminus S) \subseteq S$ $\Leftrightarrow$ $\mathbb{R}\setminus S$ is open. $\square$

Theorem 3.4.10

(a) The union of any collection of open sets is open.
(b) The intersection of any <u>finite</u> collection of open sets is open.
(c)(d) (a) and (b) remain true if we interchange "union" with "intersection", and "open" with "closed," throughout.

Proof.

(a) Let $\mathcal{U} = \bigcup_{\alpha \in A} S_\alpha$, where A is some indexing set, and each S_α is open. Let $x \in \mathcal{U}$: then $x \in S_\beta$ for some particular $\beta \in A$. Since S_β is open, $\exists \varepsilon > 0$: $N(x,\varepsilon) \subseteq S_\beta$. But $S_\beta \subseteq \mathcal{U}$, so $N(x,\varepsilon) \subseteq \mathcal{U}$. So $\mathcal{U}$ is open.

(b) Let $\mathcal{I} = S_1 \cap S_2 \cap \cdots \cap S_n$, where each S_j $(1 \le j \le n)$ is open. If $x \in \mathcal{I}$, then $x \in S_j$ for each j, so, since each S_j is open, $\exists \varepsilon_j > 0$: $N(x,\varepsilon_j) \subseteq S_j$. Let ε be the minimum of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$: then $N(x,\varepsilon) \subseteq N(x,\varepsilon_j) \subseteq S_j$ for each j, so $N(x,\varepsilon) \subseteq \mathcal{I}$. So $\mathcal{I}$ is open.

(c)(d) Proof omitted; think "complements." $\square$

Example.

Find the interior and boundary of each of these sets, and whether the set is open/closed/neither.

(i) $U = (-2,3) \cup \{4\} \cup (5,\infty)$ (ii) $V = \bigcup_{n=1}^{\infty} (n, n+1)$

Solution.

(i) $\text{int } U = (-2,3) \cup (5,\infty)$

$\text{bd } U = \{-2,3,4,5\}$

U is neither open nor closed

(ii) $\text{int } V = \emptyset$

$\text{bd } V = \mathbb{N}$

V is open