Completeness, continued.

Recall: $\mathbb{R}$ is complete (if $S \subseteq \mathbb{R}$ is nonempty and bounded above, resp. below, then $\sup S$, resp. $\inf S$, exists) $\Rightarrow$ $\mathbb{R}$ has the Archimedean property ($\mathbb{N} \subseteq \mathbb{R}$ is not bounded above) $\Rightarrow$ Thm. 3.3.10 (various properties of "$>$" on $\mathbb{R}$) $\Rightarrow$ $\mathbb{Q}$ is dense in $\mathbb{R}$ ($\forall x, y \in \mathbb{R}$ with $x<y$, $\exists q \in \mathbb{Q} : x<q<y$).

What about irrational numbers?

First of all, they exist:

Theorem 3.3.12.
If $p \in \mathbb{N}$ is prime, then $\exists x \in \mathbb{R}: x>0$ and $x^2 = p$. (Remark: we write $x = \sqrt{p}$.)

Sketch of proof. (DIY: fill in the details.)

Let $S = \{ r \in \mathbb{R} : r>0 \text{ and } r^2 < p \}$. Then S is nonempty ($1 \in S$) and bounded above (by p), so by completeness, $x = \sup S$ exists.

Claim: $x^2 = p$.
To prove this, we prove $x^2 < p$ and $x^2 > p$ are false. Then by the trichotomy law (Axiom O1, p. 114), it follows that $x^2 = p$.

We prove $x^2 < p$ is false by contradiction:

Suppose $x^2 < p$. Then $\frac{p-x^2}{2x+1} > 0$, so by Thm. 3.3.10(c), $\exists n \in \mathbb{N}$:

$$\frac{1}{n} < \frac{p-x^2}{2x+1}.$$

Now do some algebra to conclude that $(x+1/n)^2 < p$, contradicting the assumption $x = \sup S$. So $x^2 < p$ is false.

Similarly, we show that $x^2 > p$ is false. So $x^2 = p$. So $\sqrt{p}$ exists. $\square$

Next, we show:

<u>Thm. 3.3.1.</u>
If $p \in \mathbb{N}$ is prime, then $\sqrt{p} \notin \mathbb{Q}$.

<u>Proof</u> (by contradiction).
Let $p \in \mathbb{N}$ be prime; assume $\sqrt{p} = m/n$ where $m, n \in \mathbb{N}$ and m, n are coprime (their only common factor in $\mathbb{N}$ is 1). Squaring both sides gives $p = m^2/n^2$, or

$$n^2 p = m^2. \qquad (*)$$

So $p | m^2$. But m^2 and m have the same primes as factors, so $p | m$. Say $m = pk$, where $k \in \mathbb{Z}$. Then $(*)$ gives $n^2 p = p^2 k^2$: divide by p to get $n^2 = pk^2$, so $p | n^2$, so $p | n$.

So $p | m$ and $p | n$, contradicting the fact that m, n are coprime. So $\sqrt{p} \notin \mathbb{Q}$. $\square$

Finally, we have:

Theorem 3.3.15.
The irrationals are dense in $\mathbb{R}$: given $x, y \in \mathbb{R}$ with $x < y$, $\exists$ an irrational $w \in \mathbb{R}$ with

$$x < w < y.$$

Proof.
Assume $x, y \in \mathbb{R}$ with $x < y$. By Thm. 3.3.13, $\exists\, q \in \mathbb{Q}$:

$$x/\sqrt{2} < q < y/\sqrt{2}.$$

Multiply by $\sqrt{2}$:

$$x < \sqrt{2}q < y.$$

One shows that, since q is rational, $\sqrt{2}q$ is not, and we're done. $\square$

Proposition. Let S and T be nonempty subsets of $\mathbb{R}$. Suppose $\sigma \leq \tau$ for all $\sigma \in S, \tau \in T$. Then $\sup S \leq \inf T$.

Proof. Given nonempty sets $S, T \subseteq R$, suppose $\sigma \leq \tau$ for all $\sigma \in S, \tau \in T$.

We note first that, since $\sigma \leq \tau \ \forall \sigma \in S, \tau \in T$, we see that S is bounded above, by any element of T. Moreover, since $\tau \geq \sigma \ \forall \tau \in T, \sigma \in S$, we see that T is bounded below, by any element of S. So $\sup S$ and $\inf T$ exist.

Now let $\varepsilon > 0$. Since $\inf T$ is the *greatest* lower bound for T, and since $\inf T + \varepsilon$ is greater than $\inf T$, we see that $\inf T + \varepsilon$ is *not* a lower bound for T. So there exists some $\tau \in T$ with

$$\tau < \inf T + \varepsilon. \tag{$*$}$$

Further, by assumption, we have $\tau \geq \sigma$ for every $\sigma \in S$, so τ is an upper bound for S, and is therefore greater than or equal to the least upper bound for S—that is,

$$\tau \geq \sup S. \tag{$**$}$$

Combining $(*)$ with $(**)$ gives

$$\sup S \leq \tau < \inf T + \varepsilon,$$

for arbitrary $\varepsilon > 0$. But if $\sup S < \inf T + \varepsilon$ for all $\varepsilon > 0$, then $\sup S \leq \inf T$, (see Theorem 3.2.8 in the text), and we're done. $\square$