More on completeness.

Recall: $\mathbb{R}$ is the unique* complete, ordered field.

Completeness means: if $\emptyset \neq S \subseteq \mathbb{R}$ is bounded above (respectively, below) then $\sup S$ (respectively, $\inf S$) exists.

* (proof omitted)

As a consequence, $\mathbb{N}$ is unbounded in $\mathbb{R}$. This is the Archimedean property of $\mathbb{R}$, Thm. 3.3.9.

Example of an ordered field without the Archimedean property:

Let $\mathbb{F}$ = {rational functions over $\mathbb{R}$}
$= \{ p(x)/q(x) : p(x), q(x)$ are polynomials with real coefficients and no common factors except ± 1, and $q(x) \neq 0 \}$.

Define $+, -, \times, \div$ on $\mathbb{F}$ as usual, and define $>$ on $\mathbb{F}$ by

$$\frac{p(x)}{q(x)} > \frac{r(x)}{s(x)} \quad \text{if} \quad \frac{p(x)}{q(x)} - \frac{r(x)}{s(x)}$$

is >0 for x large enough.

Then $\mathbb{F}$ is an ordered field, containing $\mathbb{N}$ (think of $n \in \mathbb{N}$ as $n/1 \in \mathbb{F}$).

FACT: $\mathbb{F}$ does not satisfy the Archimedean property:
The polynomial $\frac{x}{1}$ is $> \frac{n}{1}$ $\forall n \in \mathbb{N}$!!

Other consequences of the Archimedean property:

<u>Theorem 3.3.10.</u> TFAE (the following are equivalent) to Thm. 3.3.9:

(a) $\forall z \in \mathbb{R}, \exists n \in \mathbb{N}: n > z$.
(b) $\forall x > 0$ and $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}: nx > y$.
(c) $\forall x > 0, \exists n \in \mathbb{N}: 0 < 1/n < x$.

<u>Proof</u> We'll show only Thm. 3.3.9 $\Rightarrow$ (a) and (b) $\Rightarrow$ (c). For the rest, DIY (do it yourself).

(i) By contrapositive: assume $\exists z_0 \in \mathbb{R}: \forall n \in \mathbb{N}$, $n \leq z_0$. Then z_0 is an upper bound for $\mathbb{N}$. So Thm. 3.3.9 is false.
Therefore, Thm. 3.3.9 $\Rightarrow$ (a).

(ii) Assume (b) is true. Let $y = 1$. Then given $x > 0$, $\exists n \in \mathbb{N}: nx > 1$. Also $1 > 0$ (Exercise 3(f), Sec. 3.2.) So $nx > 1 > 0$. Multiply by $1/n$:

$$x > 1/n > 0. \quad \text{(by Axiom O4, p. 114).}$$

So (b) $\Rightarrow$ (c). $\square$

<u>Theorem 3.3.13</u> (the "density" of $\mathbb{Q}$ in $\mathbb{R}$).
If $x, y \in \mathbb{R}$ and $x < y$, then $\exists q \in \mathbb{Q}$:

$$x < q < y.$$

<u>Proof.</u> We consider only $x > 0$. The other cases are similar.

Assume $x, y \in \mathbb{R}$ and $0 < x < y$. By Thm. 3.3.10(c), $\exists\, n \in \mathbb{N}: 1/n < y - x$. Multiply by n and rearrange to get $nx + 1 < ny$.

Now by Thm. 3.3.10(a), $\exists\, m \in \mathbb{N}: m > nx$. Any such m must be > 0, since $n > 0$ and $x > 0$.

Let m_0 be the smallest such m: m_0 exists by the Well-Ordering property (Axiom 3.3.1.) Since m_0 is minimal, we have $m_0 - 1 \leq nx$, so $m_0 \leq nx + 1$. In sum,

$$nx < m_0 \leq nx + 1 < ny.$$

Divide by n:

$$x < m_0/n < y.$$

So $q = m_0/n$ works.
The cases $x < 0$ and $x = 0$ are similar.

So $x, y \in \mathbb{R}$ and $x < y \Rightarrow \exists\, q \in \mathbb{Q}: x < q < y$. $\square$