The real numbers and completeness. (Sec. 3.3.)

The real numbers $\mathbb{R}$ are the unique "complete ordered field." This means $\mathbb{R}$ satisfies:

(i) various field axioms describing behavior of $+, -, \times, \div$ (text, p. 113),

(ii) various order axioms describing behavior of $<, >, \leq, \geq$ (text, p 114),

(iii) the completeness axiom, to be explained below.

Definition 1. Let $S \subseteq \mathbb{R}$.

(a) If $\exists m \in \mathbb{R}$ with $m \geq s$ $\forall s \in S$, we call m an upper bound for S, and say S is bounded above.

(b) Define lower bound and bounded below by replacing $m \geq s$ with $m \leq s$ in (a).

(c) If an upper bound m of S belongs to S, we call m the maximum of S, written $m = \max S$.

(d) The minimum of S, written $\min S$, is defined analogously.

Example 1. Let

$$S = (0,1], \quad T = \{3 - \tfrac{1}{n} : n \in \mathbb{N}\}, \quad U = \emptyset, \quad V = \mathbb{R}.$$

Then:

- $2, -\pi, -\frac{1}{10^{40}}, 0$ are lower bounds for S.
- S has no minimum
- $1 = \max S$
- T has no maximum
- $2 = \min T$
- $3, 7, 10^{1043}$ are upper bounds for T.
- U is bounded above/below by any $m \in \mathbb{R}$. $\max U$ and $\min U$ are undefined.
- V is not bounded above or below.

<u>Definition 2.</u> Let $S \subseteq \mathbb{R}$ be nonempty.

(a) If S is bounded above and m is the <u>smallest</u> upper bound for S, we write $m = \sup S$ (the supremum of S)

(b) Similarly, the infemum $\inf S$, if it exists, is the largest lower bound for S.

<u>Example 2.</u>

For S, T, U, V as in Example 1,

- $\inf S = 0$, $\sup S = \max S = 1$,
- $\inf T = \min T = 2$, $\sup T = 3$,
- $\inf V$ and $\sup V$ are undefined.

We can now state:

<u>The Completeness Axiom for $\mathbb{R}$.</u>
Every nonempty $S \subseteq \mathbb{R}$ that has a (finite) upper bound has a (finite) least upper bound.

A consequence:

Theorem (the Archimedean property of $\mathbb{R}$): The subset $\mathbb{N}$ of $\mathbb{R}$ is unbounded above (i.e. $\mathbb{N}$ has no upper bound).

Proof (by contradiction).

Assume $\mathbb{N}$ is bounded above. Then by the completeness axiom, $\exists\, x \in \mathbb{R}$ such that $x = \sup \mathbb{N}$. Since x is the <u>least</u> upper bound, $x-1$ is not an upper bound. So $\exists\, n \in \mathbb{N}$: $n > x-1$. But then $x < n+1$, so and since $n+1 \in \mathbb{N}$, x is not an upper bound for $\mathbb{N}$. Contradiction (to the claim that $x = \sup \mathbb{N}$). So $\mathbb{N}$ is not bounded above.

$\square$