Induction, concluded.

1) Divisibility.

Example 1. Show that, $\forall n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$,

$12^n - 5^n$ is a multiple of 7.

Proof

Let $A(n)$ be the above statement.

Is $A(0)$ true?

$$12^0 - 5^0 = 1 - 1 = 0,$$

which is a multiple of 7. So $A(0)$ is true.

Now assume $A(k)$:

$$12^k - 5^k = 7m$$

for some $m \in \mathbb{Z}$.

Then

$$\begin{aligned} 12^{k+1} - 5^{k+1} &= 12(12^k - 5^k) + 12 \cdot 5^k - 5^{k+1} \\ &= 12(12^k - 5^k) + 5^k(12 - 5) \\ &= 12 \cdot 7m + 5^k \cdot 7 \\ &= 7(12m + 5^k), \end{aligned}$$

so $A(k+1)$ follows.

So $A(k) \Rightarrow A(k+1)$.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

2) Other forms of induction.

(Reference: Book of Proof, Ch. 10.)

Some variations:

(a) Your starting point may not be $n = 1$.

(b) Your base step might require several cases.

(c) Your induction hypothesis and/or induction step might be more complex.

Example 2.

Define the Fibonacci numbers F_n by

$$F_1 = 1,\ F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 1, 2, \ldots).$$

(So the first few F_n's are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$$

Show that, $\forall n \in \mathbb{N}$,

$$F_n = \frac{1}{\sqrt{5}}\left(\Phi^n - (1-\Phi)^n\right),$$

where $\Phi = \frac{1}{2}(1+\sqrt{5})$.

Hint:

$$\Phi^2 = \frac{1}{4}(1+\sqrt{5})^2 = \frac{1}{4}(6+2\sqrt{5}) = \frac{1}{4}(4+2+2\sqrt{5})$$
$$= 1 + \frac{1}{2}(1+\sqrt{5}) = 1 + \Phi, \quad (*)$$

so

$$(1-\Phi)^2 = 1 - 2\Phi + \Phi^2 = 1 - 2\Phi + 1 + \Phi = 2 - \Phi. \quad (**)$$

Solution. Let $A(n)$ be the given statement.

Are $A(1)$ and $A(2)$ true?

$$1 = F_1 \overset{?}{=} \frac{1}{\sqrt{5}}(\Phi - (1-\Phi)) = \frac{1}{\sqrt{5}}(2\Phi - 1) = \frac{1}{\sqrt{5}}(1+\sqrt{5}-1) = 1,$$

by (*) and (**)

and

$$1 = F_2 \overset{?}{=} \frac{1}{\sqrt{5}}\left(\Phi^2 - (1-\Phi)^2\right) = \frac{1}{\sqrt{5}}(2\Phi - 1) = \frac{1}{\sqrt{5}}(1+\sqrt{5}-1) = 1,$$

so $A(1)$ and $A(2)$ are true.

Now assume $A(k)$ and $A(k+1)$.

Then

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ &= \frac{1}{\sqrt{5}}\left(\Phi^{k+1} - (1-\Phi)^{k+1} + \Phi^k - (1-\Phi)^k\right) \\ &= \frac{1}{\sqrt{5}}\left(\Phi^k(\Phi+1) - (1-\Phi)^k(1-\Phi+1)\right) \\ &= \frac{1}{\sqrt{5}}\left(\Phi^k(\Phi+1) - (1-\Phi)^k(2-\Phi)\right) \\ &\overset{\downarrow}{=} \frac{1}{\sqrt{5}}\left(\Phi^k \cdot \Phi^2 - (1-\Phi)^k \cdot (1-\Phi)^2\right) \\ &= \frac{1}{\sqrt{5}}\left(\Phi^{k+2} - (1-\Phi)^{k+2}\right), \end{aligned}$$

by (*) and (**)

so $A(k+2)$ follows.

So $A(k)$ and $A(k+1) \Rightarrow A(k+2)$.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

(technically, a form of strong induction.)