Quantifiers.

A) Basics.

$\forall$ (universal quantifier) means "for all" or "for each."

$\exists$ (existential quantifier) means "there exists" or "for some."

Also, we use $\ni$, or $:$, to mean "such that."

Examples:

1) $\forall n \in \mathbb{Z}$, $n = 2k$ for some $k \in \mathbb{Z}$ (if n is even) or $n = 2k+1$ for some $k \in \mathbb{Z}$ (if n is odd).

2) $\exists x \in \mathbb{R} \ni x \notin \mathbb{Q}$. That is, there is a (at least one) real number that is irrational.

We can:

(a) Combine quantifiers.

For example, we can write:

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x > y$ (which is true: take $y = x - 1$);

(ii) $\exists y \in \mathbb{R} : \forall x \in \mathbb{R}, x > y$ (which is false: no real number y is $<$ all real numbers x).

[Note: We interpret (i) as meaning a different y can work for each x. In (ii), the same y must work for all x.]

(b) Negate statements with quantifiers.

Let X be a set, and $Q(x)$ a statement about an object x. Then:

(i) $\sim(\forall x \in X, Q(x))$ is equivalent to
$\exists x \in X \ni \sim Q(x)$.

(ii) $\sim(\exists x \in X \ni Q(x))$ is equivalent to
$\forall x \in X, \sim Q(x)$.

(B) Proofs.

(1) To prove $\exists x \in X \ni Q(x)$, only one (explicit or implicit) example is required.

(2) Proving $\forall x \in X, Q(x)$ is the same as proving $x \in X \Rightarrow Q(x)$ (so it's really a $P \Rightarrow Q$ proof).

Examples:

Proposition 1.
$$\exists q \in \mathbb{Q} \ni |q - \sqrt{2}| < .01.$$

Proof
We have $\sqrt{2} = 1.4142\ldots$
Let $q = 1.41$. Then $q - \sqrt{2} = -0.0042\ldots$,
so $|q - \sqrt{2}| = 0.0042\ldots < 0.01$.
So there is a rational number within 0.01 of $\sqrt{2}$. $\square$

Proposition 2.
$$\exists p \in \{\text{prime numbers}\} \ni p > 10^{10,000,000,000}.$$

Proof
$\exists$ infinitely many primes: denote them by $p_1, p_2, p_3, \ldots$ (in ascending order). Each is larger than the previous one (since they're integers), so

$$P_{10^{10,000,000,000}} > 10^{10,000,000,000}.$$

So $\exists$ a prime number $> 10^{10,000,000,000}$. $\square$

Proposition 3.
$\forall x \in \mathbb{R}, x^2 \geqslant 6x - 9.$

Proof

Let $x \in \mathbb{R}$. Then
$(x-3)^2 \geqslant 0$ (the square of any real number is $\geqslant 0$). So

$$x^2 - 6x + 9 \geqslant 0$$
$$x^2 \geqslant 6x - 9.$$

So $x \in \mathbb{R} \Rightarrow x^2 \geqslant 6x - 9$. $\square$

Proposition 4.
$\forall \varepsilon > 0, \exists \delta > 0 : |x-3| < \delta \Rightarrow |2x-6| < \varepsilon.$

Proof.

Let $\varepsilon > 0$. Define $\delta = \varepsilon/2$. Then

$$|x-2| < \delta \Rightarrow |2x-6| = 2|x-3| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon.$$

So given $\varepsilon > 0$, there is a $\delta > 0$ such that, if $|x-2| < \delta$, then $|2x-6| < \varepsilon$. $\square$