Limit proof activity: SOLUTIONS

1. Fill in the blanks to complete the following proof.

Proposition. Let

$$f(x) = \tfrac{1}{2}x - 2.$$

Then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - 8| < \delta \Rightarrow |f(x) - 2| < \varepsilon.$$

Proof. Let $\varepsilon >$ ___0___.

[Scratchwork: Ultimately, we want to assure that $|f(x) - 2| <$ ___ε___. But

$$|f(x) - 2| = \left|\tfrac{1}{2}x - 2 - 2\right| = \left|\tfrac{1}{2}x - 4\right| = \tfrac{1}{2}\left|x - \underline{\quad 8 \quad}\right|,$$

so we want $\frac{1}{2}\left|x - 8\right| < \varepsilon$, meaning $\left|x - 8\right| <$ ___2ε___. So $\delta =$ ___2ε___ works. Then this is what we write.]

Let $\delta =$ ___2ε___. Then

$$|x - 8| < \delta \Rightarrow |f(x) - 2| = \left|\underline{\quad \tfrac{1}{2}x - 2 \quad} - 2\right| = \left|\tfrac{1}{2}x - 4\right| = \tfrac{1}{2}\left|x - 8\right| < \tfrac{1}{2} \cdot \underline{\quad 2\varepsilon \quad} = \varepsilon.$$

So $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - 8| < \delta \underline{\quad \Rightarrow \quad} |f(x) - 2| < \varepsilon.$$

□

2. Fill in the blanks.

Let's summarize what we've shown in Problem 1 above. We've shown that we can assure that $|f(x) - 2| < \varepsilon$, for any positive number ε, as long as x is within ___2ε___ units of 8. Now think of ε as being a *small* positive number. Then what we've shown is: we can make $f(x)$ as close as we want to 2 — specifically, we can assure that $f(x)$ is within ___ε___ units of 2 — as long as we choose x close enough to 8 — that is, within ___2ε___ units of 8.

So: no matter how close we want $f(x)$ to be to ___2___, we can achieve that, if we choose x close enough to ___8___. Or another words: x being close to ___8___ guarantees that $f(x)$ is close to ___2___. So what we've just proved is that

$$\lim_{x \to \underline{\ 8\ }} f(x) = \underline{\quad 2 \quad}.$$