Homework Assignment 3: Solutions to Selected Exercises

Please note that starred problems in the Exercises have hints and/or solutions at the back of the book.

Assignment from the text:

Section 3.3: Exercises 1, 2, 3adefghjkmn, 4adefghjkmn, 8, 12, 17.

Section 3.1

1. **(a)** T. **(b)** F. **(c)** F. **(d)** F. **(e)** T. **(f)** T.

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2. **(a)** T. **(b)** F. **(c)** F. **(d)** T. **(e)** T. **(f)** F. **(g)** T. **(h)** F.

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3. **(a)** $\max\{1,3\}$=3, $\sup\{1,3\} = 3$.

(d) $\max(0,4)$ does not exist; $\sup(0,4) = 4$.

(e) $\max\left\{\frac{1}{2n} : n \in \mathbb{N}\right\} = \frac{1}{2}$; $\sup\left\{\frac{1}{2n} : n \in \mathbb{N}\right\} = \frac{1}{2}$.

(f) $\max\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ does not exist; $\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$.

(g) $\max\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ does not exist; $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$.

(h) $\max\left\{(-1)^n\left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \frac{3}{2}$; $\sup\left\{(-1)^n\left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \frac{3}{2}$.

(j) $\max(-\infty,4)$ does not exist; $\sup(-\infty,4) = 4$.

(k) $\max\cup_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ does not exist; $\sup\cup_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = 2$.

(m) $\max\{r \in \mathbb{Q} : r < 5\}$ does not exist; $\sup\{r \in \mathbb{Q} : r < 5\} = 5$.

(n) $\max\{r \in \mathbb{Q} : r^2 \leq 5\}$ does not exist; $\sup\{r \in \mathbb{Q} : r^2 \leq 5\} = \sqrt{5}$.

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4. **(a)** $\min\{1,3\}$=1, $\inf\{1,3\} = 1$.

(d) $\min(0,4)$ does not exist; $\inf(0,4) = 0$.

(e) $\min\left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$ does not exist; $\inf\left\{\frac{1}{2n} : n \in \mathbb{N}\right\} = 0$.

(f) $\min\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 0$; $\inf\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 0$.

(g) $\min\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = \frac{1}{2}$; $\inf\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = \frac{1}{2}$.

(h) $\min\left\{(-1)^n\left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = -2$; $\inf\left\{(-1)^n\left(1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\} = -2$.

(j) $\min(-\infty,4)$ does not exist; $\inf(-\infty,4)$ does not exist.

(k) $\min\cup_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ does not exist; $\inf\cup_{n=1}^{\infty}\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = 0$.

(m) $\min\{r \in \mathbb{Q} : r < 5\}$ does not exist; $\inf\{r \in \mathbb{Q} : r < 5\}$ does not exist.

(n) $\min\{r \in \mathbb{Q} : r^2 \leq 5\}$ does not exist; $\inf\{r \in \mathbb{Q} : r^2 \leq 5\} = -\sqrt{5}$.

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8. See Writing Assignment #1.

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12. See Writing Assignment #1.

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17. Proof. Let $a, b \in \mathbb{N}$. Let $S = \{m \in \mathbb{N} : ma > b\}$. Then S is nonempty, by Theorem 3.3.10(b). So, by the well-ordering property (3.1.1), S has a least element m_0. Now let $q = m_0 - 1$. Then we have either $q = 0$ or $q \in \mathbb{N}$. We consider both cases:

- $q = 0$. Then $m_0 = 1$. Then, since $m_0 \in S$, we have $m_0 a = a > b$. So we can write $b = 0 \cdot a + b$; that is, $b = qa + r$ with $q = 0$ and $r = b < a$.
- $q \in \mathbb{N}$. Since $q = m_0 - 1$ and m_0 is the smallest element of S, we have $q \notin S$, so by definition of S, $qa \leq b$. Write $r = b - qa$; then $r \geq 0$. Moreover, $r < a$, because $r \geq a$ would imply $b - qa \geq a$, which would imply $b \geq (q+1)a = m_0 a$, contradicting the fact that $m_0 \in S$.

  So $r = b - qa$, which is equivalent to $b = qa + r$, where $q \in \mathbb{N}$ and r is an integer with $0 \leq r < a$.

In either case, we have $b = qa + r$, where $b, q \in \mathbb{N} \cup \{0\}$. □