Homework Assignment 2: Solutions to Selected Exercises

Please note that starred problems in the Exercises have hints and/or solutions at the back of the book.

Assignment from the text:

Section 3.1: Exercises 4, 5, 6, 7, 14, 15, 18, 19, 21.

Section 3.1

4. Proof. Let $A(n)$ be the statement

$$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1^3 &\overset{?}{=} \frac{1}{4} \cdot 1^2 \cdot (1+1)^2 \\ 1 &= 1, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 1^3 + 2^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2.$$

Then

$$\begin{aligned} 1^3 + 2^3 + \cdots + k^3 &= (1^3 + 2^3 + \cdots + k^3) + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. □

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6. Proof. Let $A(n)$ be the statement

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Step 1: Is $A(1)$ true?

$$\frac{1}{1\cdot 2} \stackrel{?}{=} \frac{1}{1+1}$$
$$\frac{1}{2} = \frac{1}{2},$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Then

$$\begin{aligned}
&\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} \\
&= \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)}\right) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

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14. Proof. Let $A(n)$ be the statement

$$9^n - 4^n \text{ is a multiple of } 5.$$

Step 1: Is $A(1)$ true?

$$9^n - 4^n = 9 - 4 = 5 = 5\cdot 1,$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 9^k - 4^k = 5m \quad \text{for some } m \in \mathbb{Z}.$$

Then

$$\begin{aligned}
9^{k+1} - 4^{k+1} &= 9(9^k - 4^k) + 9\cdot 4^k - 4^{k+1} \\
&= 9(9^k - 4^k) + 4^k(9-4) \\
&= 9(5m) + 4^k\cdot 5 = 5(9m + 4^k),
\end{aligned}$$

and $9m + 4^k \in \mathbb{Z}$, so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

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15. This is just like Exercise 14 above (and we did this in class on 9/8).

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18. Proof. Let $A(n)$ be the statement

$$2 + 5 + 8 + \cdots + (3n - 1) = \frac{1}{2}n(3n + 1).$$

Step 1: Is $A(1)$ true?

$$\begin{aligned} 2 &\stackrel{?}{=} \frac{1}{2} \cdot 1 \cdot (3 \cdot 1 + 1) \\ 2 &= 2, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 2 + 5 + 8 + \cdots + (3k - 1) = \frac{1}{2}k(3k + 1).$$

Then

$$\begin{aligned} 2 + 3 + 5 + 8 + \cdots + (3(k+1) - 1) &= \big(2 + 5 + 8 + \cdots + (3k - 1)\big) + (3(k+1) - 1) \\ &= \frac{1}{2}k(3k+1) + (3k+2) = \frac{k(3k+1) + 2(3k+2)}{2} \\ &= \frac{3k^2 + k + 6k + 4}{2} = \frac{3k^2 + 7k + 4}{2} \\ &= \frac{(k+1)(3k+4)}{2} = \frac{(k+1)(3(k+1)+1)}{2}, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

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21. Proof. Let $A(n)$ be the statement

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Step 1: Is $A(2)$ true?

$$\begin{aligned} \left(1 - \frac{1}{2^2}\right) &\stackrel{?}{=} \frac{2+1}{2 \cdot 2} \\ \frac{3}{4} &= \frac{3}{4}, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

Then

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)\\
&=\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\cdot\left(1-\frac{1}{(k+1)^2}\right)\\
&=\frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2k}\cdot\frac{(k+1)^2-1}{(k+1)^2}\\
&=\frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2}\\
&=\frac{k+1}{2k}\cdot\frac{k(k+2)}{(k+1)^2}=\frac{k+2}{2(k+1)},
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$