Homework Assignment 1: Solutions to Selected Exercises

Please note that starred problems in the Exercises have hints and/or solutions at the back of the book.

Assignment from the text:

Section 1.1: Exercises 3, 4, 9, 10, 11, 12.

Section 1.2: Exercises 1, 2, 4, 5, 7, 10, 11, 13, 14, 17, 18.

Section 1.3: Exercises 1–5, 6acefgk, 7adgh, 8.

Section 1.4: Exercises 1, 2, 3, 6, 9, 11, 17, 18, 24abc, 31.

Section 1.1

4. **(a)** The function $f(x) = x^2 - 9$ is not continuous at $x = 3$. **(b)** The relation R is neither reflexive nor symmetric. **(c)** Four and nine are not relatively prime. **(d)** x is not in A and x is in B. **(e)** There is some $x < 7$ such that $f(x)$ is in C. **(f)** There is a convergent sequence (a_n) that is either not monotone or not bounded. **(g)** There exist a continuous function f and an open set A such that $f^{-1}(A)$ is not open.

10. **(a)** False. **(b)** False. **(c)** True. **(d)** False. **(e)** True. **(f)** True. **(g)** False. **(h)** True. **(i)** False. **(j)** True.

12. **(a)** $x \perp N \wedge x \not\perp M$. **(b)** $x \not\perp M \wedge x \not\perp N$. **(c)** $x \perp N \Rightarrow x \perp M$. **(d)** $x \perp M \Rightarrow x \not\perp N$. **(e)** $\sim (x \perp M \wedge x \perp N)$.

Section 1.2:

1. **(a)** True. **(b)** False. **(c)** True.

2. **(a)** False. **(b)** True. **(c)** True.

4. **(a)** Not everyone likes Robert. **(b)** No students work part-time. **(c)** There exists a square matrix that is triangular. **(d)** $\forall x$ in B, $f(x) \leq k$. **(e)** There exists an $x < 5$ such that $f(x) \geq 3$ and $f(x) \leq 7$. **(f)** There exists an x in A such that, for all $y \in B$, $f(x) \geq f(y)$.

7. **(c)** (ii). **(d)** (i).

14. **(a)** A function f is *periodic* if $\exists k > 0 \ni \forall x, f(x + k) = f(x)$. **(b)** A function f is *not periodic* (or *aperiodic*) if $\forall k > 0, \exists x \ni f(x + k) \neq f(x)$.

18. **(a)** A function $f\colon A \to B$ is *surjective* if $\forall y \in B, \exists x \in A \ni f(x) = y$. **(b)** A function $f\colon A \to B$ is *not surjective* if $\exists y \in B \ni \forall x \in A, f(x) \neq y$.

Section 1.3:

1. **(a)** True. **(b)** False. **(c)** False. **(d)** False. **(e)** True.

2. **(a)** True. **(b)** False. **(c)** True. **(d)** True. **(e)** False.

4. **(a)** If some violets are blue, then all roses are red. **(b)** If A is not invertible, then there exists a nontrivial solution to $Ax = 0$. **(c)** If $f(C)$ is connected, then f is continuous and C is connected.

5. **(a)** If not all roses are red, then no violets are blue. **(b)** If there is no nontrivial solution to $Ax = 0$, then A is invertible. **(c)** If f is discontinuous or C is not connected, then $f(C)$ is not connected.

6. **(a)** $x = -4$. **(c)** $x = 0.3$. **(e)** $n = 41$. **(f)** The number 2 is a counterexample. **(g)** The integer 103 is a counterexample. **(k)** $x = 0$.

7. **(a)** Suppose that p is odd and q is odd. Then we can write $p = 2k + 1$ and $q = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. Then

$$p + q = 2k + 1 + 2\ell + 1 = 2(k + \ell + 1).$$

Since $k + \ell + 1 \in \mathbb{Z}$, $p + q$ is even.

So p, q odd $\Rightarrow p + q$ is even. $\square$

(d) Suppose that p is odd and q is even. Then we can write $p = 2k+1$ and $q = 2\ell$, where $k, \ell \in \mathbb{Z}$. Then

$$p + q = 2k + 1 + 2\ell = 2(k + \ell) + 1.$$

Since $k + \ell \in \mathbb{Z}$, $p + q$ is odd.

So p odd, q even $\Rightarrow p + q$ is odd. $\square$

(g) Suppose it's not the case that p is odd and q is odd. Then either p is even or q is even. In the first case, write $p = 2k$ where $k \in \mathbb{Z}$. Then

$$pq = (2k) \cdot q = 2(kq).$$

Since $kq \in \mathbb{Z}$, pq is even. In the second case, write $q = 2\ell$ where $\ell \in \mathbb{Z}$. Then

$$pq = p \cdot (2\ell) = 2(p\ell).$$

Since $p\ell \in \mathbb{Z}$, pq is even.

So pq odd $\Rightarrow p, q$ are odd. $\square$

8. Let $f(x) = 4x + 7$, and assume that $f(x_1) = f(x_2)$. Then $4x_1 + 7 = 4x_2 + 7$. Subtracting 7 from both sides, and then dividing both sides of the result by 4, we find that $x_1 = x_2$.

So, if $f(x) = 4x + 7$, then $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. $\square$

Section 1.4:

1. **(a)** True. **(b)** True. **(c)** True.

2. **(a)** True. **(b)** False. **(c)** True.

3. Let $\varepsilon > 0$. [Scratchwork: we want $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$, or equivalently $|3x + 5 - 11| < \varepsilon$. But

$$|3x + 5 - 11| = |3x - 6| = 3|x - 2|.$$

So it's enough to choose $3|x - 2| < \varepsilon$, meaning $|x - 2| < \varepsilon/3$. So this is what we write:] Let $\delta = \varepsilon/3$. Then

$$|x - 2| < \delta \Rightarrow |3x + 5 - 11| = |3x - 6| = 3|x - 2| < 3 \cdot \varepsilon/3 = \varepsilon,$$

so $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$.

So for every $\varepsilon > 0$, $\exists \delta > 0 \ni$

$$2 - \varepsilon < 7 - 5x < 2 + \varepsilon \Rightarrow 11 - \varepsilon < 3x + 5 < 11 + \varepsilon. \qquad \square$$

6. Let $x = 0$. Then for any real number y, $xy = 0 \cdot y = 0 = x$. So there exists a real number x such that for every real number y, $xy = x$. $\square$

11. Let $x > 5$. Let $y = \frac{3x}{5-x}$. Note that $y < 0$ since $x > 5$. Also note that we can solve for x to find that $x = 5y/(y + 3)$. So $\forall x > 5, \exists y < 0 \ni x = 5y/(y + 3)$. $\square$

17. **(a)** This is a proof of the converse of the given statement, not the statement itself. **(b)** This is a proof of the contrapositive of the given statement, and is therefore OK.

18. **(a)** This is OK. **(b)** This is OK.

24. **(a)** $a^2 + b^2 \neq c^2$. **(b)** $a^2 + b^2 \neq c^2$. **(c)** $2k + 3$, $2k + 3$, 15, $-3/2$. Integer.

31. $x = -1$ is a counterexample, since then $x^2 + 1 = 2$, which is not less than $(x + 1)^2 = 0$.