1.1 EXERCISES

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.
   (a) In order to be classified as a statement, a sentence must be true.
   (b) Some statements are both true and false.
   (c) When statement p is true, its negation $\sim p$ is false.
   (d) A statement and its negation may both be false.
   (e) In mathematical logic, the word "or" has an inclusive meaning.

2. Mark each statement True or False. Justify each answer.
   (a) In an implication $p \Rightarrow q$, statement p is referred to as the proposition.
   (b) The only case where $p \Rightarrow q$ is false is when p is true and q is false.
   (c) "If p, then q" is equivalent to "p whenever q."
   (d) The negation of a conjunction is the disjunction of the negations of the individual parts.
   (e) The negation of $p \Rightarrow q$ is $q \Rightarrow p$.

3. Write the negation of each statement. ☆
   (a) The 3×3 identity matrix is singular.
   (b) The function $f(x) = \sin x$ is bounded on $\mathbb{R}$.
   (c) The functions f and g are linear.
   (d) Six is prime or seven is odd.
   (e) If x is in D, then $f(x) < 5$.
   (f) If (a_n) is monotone and bounded, then (a_n) is convergent.
   (g) If f is injective, then S is finite or denumerable.

4. Write the negation of each statement.
   (a) The function $f(x) = x^2 - 9$ is continuous at $x = 3$.
   (b) The relation R is reflexive or symmetric.
   (c) Four and nine are relatively prime.
   (d) x is in A or x is not in B.
   (e) If $x < 7$, then $f(x)$ is not in C.
   (f) If (a_n) is convergent, then (a_n) is monotone and bounded.
   (g) If f is continuous and A is open, then $f^{-1}(A)$ is open.

5. Identify the antecedent and the consequent in each statement. ☆
   (a) M has a zero eigenvalue whenever M is singular.
   (b) Linearity is a sufficient condition for continuity.
   (c) A sequence is Cauchy only if it is bounded.
   (d) $x < 3$ provided that $y > 5$.

6. Identify the antecedent and the consequent in each statement.
   (a) A sequence is convergent if it is Cauchy.
   (b) Convergence is a necessary condition for boundedness.
   (c) Orthogonality implies invertability.
   (d) K is closed and bounded only if K is compact.

7. Construct a truth table for each statement.
   (a) $p \Rightarrow \sim q$ ✫
   (b) $[p \wedge (p \Rightarrow q)] \Rightarrow q$
   (c) $[p \Rightarrow (q \wedge \sim q)] \Leftrightarrow \sim p$ ✫

8. Construct a truth table for each statement.
   (a) $p \vee \sim q$
   (b) $p \wedge \sim p$
   (c) $[(\sim q) \wedge (p \Rightarrow q)] \Rightarrow \sim p$

9. Indicate whether each statement is True or False. ✫
   (a) $3 \leq 5$ and 11 is odd.
   (b) $3^2 = 8$ or $2 + 3 = 5$.
   (c) $5 > 8$ or 3 is even.
   (d) If 6 is even, then 9 is odd.
   (e) If $8 < 3$, then $2^2 = 5$.
   (f) If 7 is odd, then 10 is prime.
   (g) If 8 is even and 5 is not prime, then $4 < 7$.
   (h) If 3 is odd or $4 > 6$, then $9 \leq 5$.
   (i) If both $5 - 3 = 2$ and $5 + 3 = 2$, then $9 = 4$.
   (j) It is not the case that 5 is even or 7 is prime.

10. Indicate whether each statement is True or False.
   (a) $2 + 3 = 5$ and 5 is even.
   (b) $3 + 4 = 5$ or $4 + 5 = 6$.
   (c) 7 is even or 6 is not prime.
   (d) If $4 + 4 = 8$, then 9 is prime.
   (e) If 6 is prime, then $8 < 6$.
   (f) If $6 < 2$, then $4 + 4 = 8$.
   (g) If 8 is prime or 7 is odd, then 9 is even.
   (h) If $2 + 5 = 7$ only if $3 + 4 = 8$, then $3^2 = 9$.
   (i) If both $5 - 3 = 2$ and $5 + 3 = 8$, then $8 - 3 = 4$.
   (j) It is not the case that 5 is not prime and 3 is odd.

11. Let p be the statement "The figure is a polygon," and let q be the statement "The figure is a circle." Express each of the following statements in symbols. ✫
   (a) The figure is a polygon, but it is not a circle.
   (b) The figure is a polygon or a circle, but not both.
   (c) If the figure is not a circle, then it is a polygon.
   (d) The figure is a circle whenever it is not a polygon.
   (e) The figure is a polygon iff it is not a circle.

12. Let m be the statement "x is perpendicular to M," and let n be the statement "x is perpendicular to N." Express each of the following statements in symbols.
   (a) x is perpendicular to N but not perpendicular to M.
   (b) x is not perpendicular to M, nor is it perpendicular to N.

(c) x is perpendicular to N only if x is perpendicular to M.
(d) x is not perpendicular to N provided it is perpendicular to M.
(e) It is not the case that x is perpendicular to M and perpendicular to N.

13. Define a new sentential connective ∇, called *nor*, by the following truth table.

p	q	$p \nabla q$
T	T	F
T	F	F
F	T	F
F	F	T

(a) Use a truth table to show that $p \nabla p$ is logically equivalent to $\sim p$.
(b) Complete a truth table for $(p \nabla p) \nabla (q \nabla q)$.
(c) Which of our basic connectives ($p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$) is logically equivalent to $(p \nabla p) \nabla (q \nabla q)$?

14. Use truth tables to verify that each of the following is a tautology. Parts (a) and (b) are called *commutative laws*, parts (c) and (d) are *associative laws*, and parts (e) and (f) are *distributive laws*.

(a) $(p \wedge q) \Leftrightarrow (q \wedge p)$
(b) $(p \vee q) \Leftrightarrow (q \vee p)$
(c) $[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r]$
(d) $[p \vee (q \vee r)] \Leftrightarrow [(p \vee q) \vee r]$
(e) $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$
(f) $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$

Section 1.2 QUANTIFIERS

In Section 1.1 we found that the sentence

$$x^2 - 5x + 6 = 0$$

needed to be considered within a particular context in order to become a statement. When a sentence involves a variable such as x, it is customary to use functional notation when referring to it. Thus we write

$$p(x): x^2 - 5x + 6 = 0$$

1.2 EXERCISES

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.
(a) The symbol "$\forall$" means "for every."
(b) The negation of a universal statement is another universal statement.
(c) The symbol "$\ni$" is read "such that."

2. Mark each statement True or False. Justify each answer.
(a) The symbol "$\exists$" means "there exist several."
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
(c) The order in which quantifiers are used affects the truth value.

3. Write the negation of each statement. ☆
(a) Some pencils are red.
(b) All chairs have four legs.
(c) No one on the basketball team is over 6 feet 4 inches tall.
(d) $\exists\, x > 2 \ni f(x) = 7$.
(e) $\forall\, x$ in A, $\exists\, y > 2 \ni 0 < f(y) < f(x)$.
(f) If $x > 3$, then $\exists\, \varepsilon > 0 \ni x^2 > 9 + \varepsilon$.

4. Write the negation of each statement.
(a) Everyone likes Robert.
(b) Some students work part-time.
(c) No square matrices are triangular.
(d) $\exists\, x$ in $B \ni f(x) > k$.
(e) If $x > 5$, then $f(x) < 3$ or $f(x) > 7$.
(f) If x is in A, then $\exists\, y$ in $B \ni f(x) < f(y)$.

5. Determine the truth value of each statement, assuming x is a real number. Justify your answer. ☆
(a) $\exists\, x$ in the interval $[2, 4] \ni x < 7$.
(b) $\forall\, x$ in the interval $[2, 4]$, $x < 7$.
(c) $\exists\, x \ni x^2 = 5$.
(d) $\forall\, x$, $x^2 = 5$.
(e) $\exists\, x \ni x^2 \neq -3$.
(f) $\forall\, x$, $x^2 \neq -3$.
(g) $\exists\, x \ni x \div x = 1$
(h) $\forall\, x$, $x \div x = 1$.

6. Determine the truth value of each statement, assuming x is a real number. Justify your answer.
(a) $\exists\, x$ in the interval $[3, 5] \ni x \geq 4$.
(b) $\forall\, x$ in the interval $[3, 5]$, $x \geq 4$
(c) $\exists\, x \ni x^2 \neq 3$.
(d) $\forall\, x$, $x^2 \neq 3$.

(e) $\exists x \ni x^2 = -5$.
(f) $\forall x, x^2 = -5$.
(g) $\exists x \ni x - x = 0$.
(h) $\forall x, x - x = 0$.

7. Below are two strategies for determining the truth value of a statement involving a positive number x and another statement $P(x)$.

(i) Find some $x > 0$ such that $P(x)$ is true.

(ii) Let x be the name for any number greater than 0 and show $P(x)$ is true.

For each statement below, indicate which strategy is more appropriate.
(a) $\forall x > 0, P(x)$. ☆
(b) $\exists x > 0 \ni P(x)$. ☆
(c) $\exists x > 0 \ni \sim P(x)$.
(d) $\forall x > 0, \sim P(x)$.

8. Which of the following best identifies f as a constant function, where x and y are real numbers.
(a) $\exists x \ni \forall y, f(x) = y$.
(b) $\forall x \exists y \ni f(x) = y$.
(c) $\exists y \ni \forall x, f(x) = y$.
(d) $\forall y \exists x \ni f(x) = y$.

9. Determine the truth value of each statement, assuming that x and y are real numbers. Justify your answer. ☆
(a) $\forall x$ and $y, x \leq y$.
(b) $\exists x$ and $y \ni x \leq y$.
(c) $\forall x, \exists y \ni x \leq y$.
(d) $\exists x \ni \forall y, x \leq y$.

10. Determine the truth value of each statement, assuming that x and y are real numbers. Justify your answer.
(a) $\forall x, \exists y \ni xy = 0$.
(b) $\forall x, \exists y \ni xy = 1$.
(c) $\exists y \ni \forall x, xy = 1$.
(d) $\forall x, \exists y \ni xy = x$.

11. Determine the truth value of each statement, assuming that x, y, and z are real numbers. Justify your answer. ☆
(a) $\exists x \ni \forall y \exists z \ni x + y = z$.
(b) $\exists x$ and $y \ni \forall z, x + y = z$.
(c) $\forall x$ and $y, \exists z \ni y - z = x$.
(d) $\forall x$ and $y, \exists z \ni xz = y$.
(e) $\exists x \ni \forall y$ and z, $z > y$ implies that $z > x + y$.
(f) $\forall x, \exists y$ and $z \ni z > y$ implies that $z > x + y$.

12. Determine the truth value of each statement, assuming that x, y, and z are real numbers. Justify your answer.
(a) $\forall\, x$ and y, $\exists\, z \ni x + y = z$.
(b) $\forall\, x\, \exists\, y \ni \forall\, z,\ x + y = z$.
(c) $\exists\, x \ni \forall\, y, \exists\, z \ni xz = y$.
(d) $\forall\, x$ and y, $\exists\, z \ni yz = x$.
(e) $\forall\, x\, \exists\, y \ni \forall\, z,\ z > y$ implies that $z > x + y$.
(f) $\forall\, x$ and y, $\exists\, z \ni z > y$ implies that $z > x + y$.

Exercises 13 to 21 give certain properties of functions that we shall encounter later in the text. You are to do two things: (a) rewrite the defining conditions in logical symbolism using $\forall$, $\exists$, $\ni$, and $\Rightarrow$, as appropriate; and (b) write the negation of part (a) using the same symbolism. It is not necessary that you understand precisely what each term means.

Example: A function f is odd if for every x, $f(-x) = -f(x)$.

(a) defining condition: $\forall\, x,\ f(-x) = -f(x)$.
(b) negation: $\exists\, x \ni f(-x) \neq -f(x)$.

13. A function f is *even* if for every x, $f(-x) = f(x)$. ☆

14. A function f is *periodic* if there exists a $k > 0$ such that for every x, $f(x + k) = f(x)$.

15. A function f is *increasing* if for every x and y, if $x \le y$, then $f(x) \le f(y)$. ☆

16. A function f is *strictly decreasing* if for every x and y, if $x < y$, then $f(x) > f(y)$.

17. A function $f: A \to B$ is *injective* if for every x and y in A, if $f(x) = f(y)$, then $x = y$. ☆

18. A function $f: A \to B$ is *surjective* if for every y in B there exists an x in A such that $f(x) = y$.

19. A function $f: D \to R$ is *continuous* at $c \in D$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. ☆

20. A function f is *uniformly continuous on a set S* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever x and y are in S and $|x - y| < \delta$.

21. The real number L is the *limit* of the function $f: D \to R$ at the point c if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$. ☆

1.3 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.
(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to p as the antecedent.
(b) The contrapositive of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
(c) The inverse of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.
(d) To prove "$\forall\ n, p(n)$" is true, it takes only one example.
(e) To prove "$\exists\ n \ni p(n)$" is true, it takes only one example.

2. Mark each statement True or False. Justify each answer.
(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to q as the conclusion.
(b) A statement that is always false is called a lie.
(c) The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
(d) To prove "$\forall\ n, p(n)$" is false, it takes only one counterexample.
(e) To prove "$\exists\ n \ni p(n)$" is false, it takes only one counterexample.

3. Write the contrapositive of each implication. ☆
(a) If all roses are red, then some violets are blue.
(b) A is not invertible if there exists a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.
(c) If f is continuous and C is connected, then $f(C)$ is connected.

4. Write the converse of each implication in Exercise 3.

5. Write the inverse of each implication in Exercise 3.

6. Provide a counterexample for each statement.
(a) For every real number x, if $x^2 > 9$ then $x > 3$.
(b) For every integer n, we have $n^3 \geq n$.
(c) For all real numbers $x \geq 0$, we have $x^2 \leq x^3$.
(d) Every triangle is a right triangle.
(e) For every positive integer n, $n^2 + n + 41$ is prime.
(f) Every prime is an odd number.
(g) No integer greater than 100 is prime.
(h) $3^n + 2$ is prime for all positive integers n.
(i) For every integer $n > 3$, $3n$ is divisible by 6.
(j) If x and y are unequal positive integers and xy is a perfect square, then x and y are perfect squares.
(k) For every real number x, there exists a real number y such that $xy = 2$.
(l) The reciprocal of a real number $x \geq 1$ is a real number y such that $0 < y < 1$.
(m) No rational number satisfies the equation $x^3 + (x-1)^2 = x^2 + 1$.
(n) No rational number satisfies the equation $x^4 + (1/x) - \sqrt{x+1} = 0$.

***7.** Suppose p and q are integers. Recall that an integer m is even iff $m = 2k$ for some integer k and m is odd iff $m = 2k + 1$ for some integer k.

Prove the following. [You may use the fact that the sum of integers and the product of integers are again integers.]

(a) If p is odd and q is odd, then $p + q$ is even.
(b) If p is odd and q is odd, then pq is odd.
(c) If p is odd and q is odd, then $p + 3q$ is even.
(d) If p is odd and q is even, then $p + q$ is odd.
(e) If p is even and q is even, then $p + q$ is even.
(f) If p is even or q is even, then pq is even.
(g) If pq is odd, then p is odd and q is odd.
(h) If p^2 is even, then p is even. ☆
(i) If p^2 is odd, then p is odd.

8. Let f be the function given by $f(x) = 4x + 7$. Use the contrapositive implication to prove the statement: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

9. In each part, a list of hypotheses is given. These hypotheses are assumed to be true. Using tautologies from Example 1.3.12, you are to establish the desired conclusion. Indicate which tautology you are using to justify each step. ☆

(a) Hypotheses: $r \Rightarrow \sim s$, $t \Rightarrow s$
Conclusion: $r \Rightarrow \sim t$
(b) Hypotheses: r, $\sim t$, $(r \wedge s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s$, $\sim r \Rightarrow \sim t$, $\sim t \Rightarrow u$, $v \Rightarrow s$
Conclusion: $\sim v \vee u$

10. Repeat Exercise 9 for the following hypotheses and conclusions.

(a) Hypotheses: $\sim r$, $(\sim r \wedge s) \Rightarrow r$
Conclusion: $\sim s$
(b) Hypotheses: $\sim t$, $(r \vee s) \Rightarrow t$
Conclusion: $\sim s$
(c) Hypotheses: $r \Rightarrow \sim s$, $t \Rightarrow u$, $s \vee t$
Conclusion: $\sim r \vee u$

11. Assume that the following two hypotheses are true: (1) If the basketball center is healthy or the point guard is playing well, then the team will win and the fans will be happy; and (2) if the fans are happy or the coach is a millionaire, then the college will balance the budget. Derive the following conclusion: If the basketball center is healthy, then the college will balance the budget. Using letters to represent the simple statements, write out a formal proof in the format of Exercise 9.

Section 1.4 TECHNIQUES OF PROOF: II

Mathematical theorems and proofs do not occur in isolation, but always in the context of some mathematical system. For example, in Section 1.3 when we discussed a conjecture related to prime numbers, the natural context of that discussion was the positive integers. In Example 1.3.7 when

1.4 EXERCISES

*Exercises marked with * are used in later sections, and exercises marked with ✫ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.
 (a) To prove a universal statement $\forall\, x,\ p(x)$, we let x represent an arbitrary member from the system under consideration and show that $p(x)$ is true.
 (b) To prove an existential statement $\exists\, x \ni p(x)$, we must find a particular x in the system for which $p(x)$ is true.
 (c) In writing a proof, it is important to include all the logical steps.

2. Mark each statement True or False. Justify each answer.
 (a) A proof by contradiction may use the tautology $(\sim p \Rightarrow c) \Leftrightarrow p$.
 (b) A proof by contradiction may use the tautology $[(p \vee \sim q) \Rightarrow c] \Leftrightarrow (p \Rightarrow q)$.
 (c) Definitions often play an important role in proofs.

3. Prove: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $2 - \delta < x < 2 + \delta$ implies that $11 - \varepsilon < 3x + 5 < 11 + \varepsilon$.

4. Prove: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $1 - \delta < x < 1 + \delta$ implies that $2 - \varepsilon < 7 - 5x < 2 + \varepsilon$.

5. Prove: There exists a real number x such that for every real number y, we have $xy = y$.

6. Prove: There exists a real number x such that for every real number y, we have $xy = x$.

7. Prove: If n is an integer, then $n^2 + n^3$ is an even number.

8. Prove: If n is odd, then $n^2 = 8k + 1$ for some integer k.

9. Prove: There exists an integer n such that $n^2 + 3n/2 = 1$. Is this integer unique? ✫

10. Prove: There exists a rational number x such that $x^2 + 3x/2 = 1$. Is this rational number unique?

11. Prove: For every real number $x > 5$, there exists a real number $y < 0$ such that $x = 5y/(y + 3)$. ✫

12. Prove: For every real number $x > 1$, there exist two distinct positive real numbers y and z such that

$$x = \frac{y^2 + 9}{6y} = \frac{z^2 + 9}{6z}.$$

13. Prove: If $x^2 + x - 6 \geq 0$, then $x \leq -3$ or $x \geq 2$. ✫

14. Prove: If $x/(x - 2) \leq 3$, then $x < 2$ or $x \geq 3$.

15. Prove: $\log_2 7$ is irrational. ☆

16. Prove: If x is a real number, then $|x+1| \le 3$ implies that $-4 \le x \le 2$.

17. Consider the following theorem: "If m^2 is odd, then m is odd." Indicate what, if anything, is wrong with each of the following "proofs."

(a) Suppose m is odd. Then $m = 2k + 1$ for some integer k. Thus $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Thus if m^2 is odd, then m is odd.

(b) Suppose m is not odd. Then m is even and $m = 2k$ for some integer k. Thus $m^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. Thus if m is not odd, then m^2 is not odd. It follows that if m^2 is odd, then m is odd.

18. Consider the following theorem: "If $xy = 0$, then $x = 0$ or $y = 0$." Indicate what, if anything, is wrong with each of the following "proofs."

(a) Suppose $xy = 0$ and $x \neq 0$. Then dividing both sides of the first equation by x we have $y = 0$. Thus if $xy = 0$, then $x = 0$ or $y = 0$.

(b) There are two cases to consider. First suppose that $x = 0$. Then $x \cdot y = 0 \cdot y = 0$. Similarly, suppose that $y = 0$. Then $x \cdot y = x \cdot 0 = 0$. In either case, $x \cdot y = 0$. Thus if $xy = 0$, then $x = 0$ or $y = 0$.

19. Suppose x and y are real numbers. Recall that a real number m is rational iff $m = p/q$, where p and q are integers and $q \neq 0$. If a real number is not rational, then it is irrational. Prove the following. [You may use the fact that the sum of integers and the product of integers are again integers.]

(a) If x is rational and y is rational, then $x + y$ is rational.

(b) If x is rational and y is rational, then xy is rational.

(c) If x is rational and y is irrational, then $x + y$ is irrational. ☆

20. Suppose x and y are real numbers. Prove or give a counterexample. [See the definitions in Exercise 19.]

(a) If x is irrational and y is irrational, then $x + y$ is irrational.

(b) If $x + y$ is irrational, then x is irrational or y is irrational.

(c) If x is irrational and y is irrational, then xy is irrational.

(d) If xy is irrational, then x is irrational or y is irrational.

21. Consider the following theorem and proof.

Theorem: If x is rational and y is irrational, then xy is irrational.

Proof: Suppose x is rational and y is irrational. If xy is rational, then we have $x = p/q$ and $xy = m/n$ for some integers p, q, m, and n, with $q \neq 0$ and $n \neq 0$. It follows that

$$y = \frac{xy}{x} = \frac{m/n}{p/q} = \frac{mq}{np}.$$

This implies that y is rational, a contradiction. We conclude that xy must be irrational. ■

(a) Find a specific counterexample to show that the theorem is false.
(b) Explain what is wrong with the proof.
(c) What additional condition on x in the hypothesis would make the conclusion true?

22. Prove or give a counterexample: If x is irrational, then $\sqrt{x}$ is irrational.

23. Prove or give a counterexample: There do not exist three consecutive even integers a, b, and c such that $a^2 + b^2 = c^2$. ☆

24. Consider the following theorem: There do not exist three consecutive odd integers a, b, and c such that $a^2 + b^2 = c^2$.

(a) Complete the following restatement of the theorem: For every three consecutive odd integers a, b, and c, ____________.

(b) Change the sentence in part (a) into an implication $p \Rightarrow q$: If a, b, and c are consecutive odd integers, then ____________.

(c) Fill in the blanks in the following proof of the theorem.

Proof: Let a, b, and c be consecutive odd integers. Then $a = 2k + 1$, $b =$ ____________, and $c = 2k + 5$ for some integer k. Suppose $a^2 + b^2 = c^2$. Then $(2k + 1)^2 + ($________$)^2 = (2k + 5)^2$.

It follows that $8k^2 + 16k + 10 = 4k^2 + 20k + 25$ and $4k^2 - 4k -$ ____________ $= 0$. Thus $k = 5/2$ or $k =$ ____________. This contradicts k being an ____________. Therefore, there do not exist three consecutive odd integers a, b, and c such that $a^2 + b^2 = c^2$. ■

(d) Which of the tautologies in Example 1.3.12 best describes the structure of the proof?

25. Prove or give a counterexample: The sum of any five consecutive integers is divisible by five.

26. Prove or give a counterexample: The sum of any four consecutive integers is never divisible by four.

27. Prove or give a counterexample: For every positive integer n, $n^2 + 3n + 8$ is even. ☆

28. Prove or give a counterexample: For every positive integer n, $n^2 + 4n + 8$ is even.

29. Prove or give a counterexample: There do not exist irrational numbers x and y such that x^y is rational. ☆

30. Prove or give a counterexample: There do not exist rational numbers x and y such that x^y is a positive integer and y^x is a negative integer.

31. Prove or give a counterexample: For all $x > 0$, we have $x^2 + 1 < (x + 1)^2 \leq 2(x^2 + 1)$.