Section 4.1: The topology of $\mathbb{R}$

1. For this exercise, it will be helpful to recall that, for a set $S \subseteq \mathbb{R}$, S' denotes the set of *accumulation points* of S, and that an *accumulation point* x of S is an $x \in \mathbb{R}$ such that $N^*(x, \varepsilon)$ intersects S for any $\varepsilon > 0$. In other words,

$$x \in S' \Leftrightarrow \forall \varepsilon > 0, N^*(x, \varepsilon) \cap S \neq \emptyset. \quad \text{(A)}$$

Also, recall that the closure $\operatorname{cl} S$ of a set S is defined by $\operatorname{cl} S = S \cup S'$.

By filling in the blanks, we will now prove

Theorem 3.4.17(b). Let $S \subseteq \mathbb{R}$. Then $\operatorname{cl} S$ is a closed set.

Proof. To show that $\operatorname{cl} S$ is closed, it suffices to show that $\mathbb{R}\backslash(\operatorname{cl} S)$ is __open__ , and to show that $\mathbb{R}\backslash(\operatorname{cl} S)$ is open, it suffices to show (by definition of open set) that

$$x \in \mathbb{R}\backslash(\operatorname{cl} S) \Rightarrow \exists \varepsilon > 0 : N(x, \varepsilon) \subseteq \underline{\quad \mathbb{R}\backslash(\operatorname{cl} S) \quad}. \quad (*)$$

So let's prove $(*)$, and we'll be done.

Let $x \in \mathbb{R}\backslash(\operatorname{cl} S)$. Note that

$$\mathbb{R}\backslash(\operatorname{cl} S) = \mathbb{R}\backslash(S \cup S') = (\mathbb{R}\backslash S) \cap (\mathbb{R}\backslash S'), \quad \text{(CC)}$$

since the complement of a union equals the corresponding __intersection__ of the complements. So $x \in \mathbb{R}\backslash S$ and $x \in$ __$\mathbb{R}\backslash S'$__.

Since $x \in \mathbb{R}\backslash S'$, x is *not* an accumulation point of __S__ , so by equation (A), there is *some* $\varepsilon > 0$ such that $N^*(x, \varepsilon) \cap S =$ __$\emptyset$__. For such ε, then, $N^*(x, \varepsilon)$ is entirely outside of S, meaning $N^*(x, \varepsilon) \subseteq$ __$\mathbb{R}\backslash S$__. Further, since $x \in \mathbb{R}\backslash S$ as well, we in fact see that $N(x, \varepsilon) \subseteq$ __$\mathbb{R}\backslash S$__.

If we can show that $N(x, \varepsilon) \subseteq$ __$\mathbb{R}\backslash S'$__ as well, then by equation (CC), we'll conclude that $N(x, \varepsilon) \subseteq$ __$\mathbb{R}\backslash(\operatorname{cl} S)$__, and we'll be done. So let's show that $N(x, \varepsilon) \subseteq$ __$\mathbb{R}\backslash S'$__, as follows. Let $y \in N(x, \varepsilon)$. We wish to show that $y \in \mathbb{R}\backslash S'$, meaning $y \notin$ __S'__, meaning y is *not* an __accumulation__ point of S. To do this we need only, by (A), find a number $\delta > 0$ such that $N^*(y, \delta) \cap S =$ __$\emptyset$__.

To find such a δ note that, since $N(x, \varepsilon)$ is open, there is a number $\delta > 0$ such that $N(y, \delta) \subseteq$ __$N(x, \varepsilon)$__. But clearly $N^*(y, \delta) \subseteq N(y, \delta)$, moreover, we showed above that $N(x, \varepsilon) \subseteq \mathbb{R}\backslash S$. Putting this all together gives

$$N^*(y, \delta) \underline{\quad \subseteq \quad} N(y, \delta) \underline{\quad \subseteq \quad} N(x, \varepsilon) \underline{\quad \subseteq \quad} \mathbb{R}\backslash S,$$

so $N^*(y, \delta) \subseteq \mathbb{R}\backslash S$, so $N^*(y, \delta) \cap S =$ __$\emptyset$__, and we're done. □

2. Find the interior, boundary, accumulation points, isolated points, and closure of each of the following sets. Then state whether the given set is open or closed or neither, and whether the given set is compact. You don't need to justify your answers.

(a) $A = [-5, 1) \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$.

$\operatorname{int} A =$ $(-5, 1)$

$\operatorname{bd} A =$ $\{-5, 1, 2\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

$A' =$ $[-5, 1] \cup \{2\}$

$A \setminus A' =$ $\{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

$\operatorname{cl} A =$ $[-5, 1] \cup \{2\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\}$

open/closed/neither? neither

compact? no

(b) The set $\mathbb{Q}^+$ of positive rational numbers.

$\operatorname{int} \mathbb{Q}^+ =$ $\emptyset$

$\operatorname{bd} \mathbb{Q}^+ =$ $[0, \infty)$

$(\mathbb{Q}^+)' =$ $[0, \infty)$

$\mathbb{Q}^+ \setminus (\mathbb{Q}^+)' =$ $\emptyset$

$\operatorname{cl} \mathbb{Q}^+ =$ $[0, \infty)$

open/closed/neither? neither

compact? no

3. Show that the interval $I = (0, \infty)$ is not compact, as follows (pretend you didn't know the Heine-Borel Theorem).

(a) Explain carefully why the collection $\mathcal{C} = \{(0, n) \colon n \in \mathbb{N}\}$ is an open cover of I (that is, the union of the sets in $\mathcal{C}$ contains I).

Let $x \in (0, \infty)$. By Theorem 3.3.10(a), There is an $n_0 \in \mathbb{N}$ such that $n_0 > x$. But then $0 < x < n_0$, so $x \in (0, n_0)$. But then x is certainly in the union $\cup_{n \in \mathbb{N}}(0, n)$.

(b) Explain carefully why there is no finite subset of $\mathcal{C}$ whose union contains I. You may use the fact that every finite subset of $\mathbb{N}$ has an upper bound.

Let $\mathcal{B}$ be a finite subset of $\mathcal{C}$. Write

$$\mathcal{B} = \{(0, n_1), (0, n_2), \ldots, (0, n_k)\},$$

for some positive integer k. Let M be any upper bound for the set $\{n_1, n_2, \ldots, n_k\}$. Then $M + 1$ is clearly not in any of the intervals in $\mathcal{B}$, so $M + 1$ is not in the union of these intervals. So we have found a cover (namely, $\mathcal{C}$) of $(0, \infty)$ with no finite subcover. So $(0, \infty)$ is not compact.