3.1 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.

(a) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \geq k$ for all $k \in S$.

(b) The principle of mathematical induction enables us to prove that a statement is true for all natural numbers without directly verifying it for each number.

2. Mark each statement True or False. Justify each answer.

(a) A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction hypothesis.

(b) Suppose m is a natural number greater than 1. To prove $P(k)$ is true for all $k \geq m$, we must first show that $P(k)$ is false for all k such that $1 \leq k < m$.

***3.** Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

*4. Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

5. Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$. ☆

*6. Prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \text{ for all } n \in \mathbb{N}.$$

*7. Prove that $1 + r + r^2 + \cdots + r^n = (1 - r^{n+1})/(1 - r)$ for all $n \in \mathbb{N}$, when $r \neq 1$. ☆

*8. Prove that

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}, \text{ for all } n \in \mathbb{N}.$$

9. Prove that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, for all $n \in \mathbb{N}$.

10. Prove that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$, for all $n \in \mathbb{N}$.

11. Prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}, \text{ for all } n \in \mathbb{N}.$$

12. Prove that $1 + 2 \cdot 2 + 3 \cdot 2^2 + \cdots + n2^{n-1} = (n-1)2^n + 1$, for all $n \in \mathbb{N}$.

13. Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$. ☆

14. Prove that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

15. Prove that $12^n - 5^n$ is a multiple of 7 for all $n \in \mathbb{N}$.

16. If a, b, and $c \in \mathbb{N}$ such that $a - b$ is a multiple of c, prove that $a^n - b^n$ is a multiple of c for all $n \in \mathbb{N}$.

17. Indicate what is wrong with each of the following induction "proofs."

(a) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "Any collection of n marbles consists of marbles of the same color." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Clearly, $P(1)$ is a true statement. Now suppose that $P(k)$ is a true statement for some $k \in \mathbb{N}$. Let S be a collection of $k + 1$ marbles. If one marble, call it x, is removed, then the induction hypothesis applied to the remaining k marbles implies that these k marbles all have the same color. Call this color C. Now if x is returned to the set S and a different marble is removed, then again the remaining k marbles must all be of the same color C. But one of these mar-

(b) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "$n^2 + 7n + 3$ is an even integer." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose that $P(k)$ is true for some $k \in \mathbb{N}$. That is, $k^2 + 7k + 3$ is an even integer. But then

$$\begin{aligned}(k + 1)^2 + 7(k + 1) + 3 &= (k^2 + 2k + 1) + 7k + 7 + 3 \\ &= (k^2 + 7k + 3) + 2(k + 4),\end{aligned}$$

and this number is even, since it is the sum of two even numbers. Thus $P(k + 1)$ is true. We conclude by induction that $P(n)$ is true for all $n \in \mathbb{N}$. ■

18. Prove that $2 + 5 + 8 + \cdots + (3n - 1) = \frac{1}{2}n(3n + 1)$ for all $n \in \mathbb{N}$.

19. Conjecture a formula for the sum $5 + 9 + 13 + \cdots + (4n + 1)$, and prove your conjecture using mathematical induction. ☆

20. Prove that

$$(2)(6)(10)(14)\cdots(4n - 2) = \frac{(2n)!}{n!}, \text{ for all } n \in \mathbb{N}.$$

21. Prove that $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n + 1}{2n}$, for all $n \in \mathbb{N}$ with $n \geq 2$.

22. Prove that $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$, for all $n \in \mathbb{N}$, where $i = \sqrt{-1}$. You may use the identities $\cos(a + b) = \cos a \cos b - \sin a \sin b$ and $\sin(a + b) = \sin a \cos b + \cos a \sin b$.

23. Indicate for which natural numbers n the given inequality is true. Prove your answers by induction.

(a) $n^2 \leq n!$ ☆
(b) $n^2 \leq 2^n$
(c) $2^n \leq n!$

***24.** Use induction to prove Bernoulli's inequality: If $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

25. Prove that for all integers $x \geq 8$, x can be written in the form $3m + 5n$, where m and n are nonnegative integers. ☆

26. Consider the statement "For all integers $x \geq k$, x can be written in the form $5m + 7n$, where m and n are nonnegative integers."

(a) Find the smallest value of k that makes the statement true.
(b) Prove the statement is true with k as in part (a).