Minimal dynamical systems and C*-algebras

Shen Lu

CU Boulder

MSU Denver
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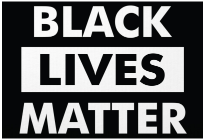
Mathematics
UNIVERSITY OF COLORADO **BOULDER**



Outline

1. Motivation
2. Dynamical systems
   - irrational rotation on the circle
   - odometer action on the Cantor set
3. C*-algebras by two examples
   - continuous functions on [0, 1]
   - $n \times n$ matrices
4. Crossed product C*-algebras
5. My current research

Motivation

Dynamical systems arise naturally: movement inside a mechanical watch, weather patterns in climate modelling, stirring a cup of trail mix, etc.

Mathematical formalization: Let X be a set (e.g. cup of trail mix, the real number line, or a circle) and $\varphi : X \to X$ be a bijective map (1-to-1 and onto) with inverse φ^{-1}. Both φ and φ^{-1} can be iterated:

$$\varphi^2 = \varphi \circ \varphi, \quad \varphi^3 = \varphi \circ \varphi \circ \varphi, \text{ etc.},$$
$$\varphi^{-2} = \varphi^{-1} \circ \varphi^{-1}, \quad \varphi^{-3} = \varphi^{-1} \circ \varphi^{-1} \circ \varphi^{-1}, \text{ etc.},$$

$$\text{and we set } \varphi^0 = \mathrm{Id}_X, \text{ i.e., } \varphi^0(x) = x.$$

For each $x \in X$ and integer n, $\varphi^n(x)$ gives the location of x in X after n iterations of φ.

We will exclusively consider when X is a compact metric space and φ is a homeomorphism.

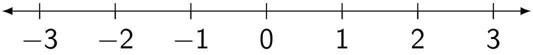
Definitions and questions

Given a dynamical system, (X, φ) with iterations φ^n for all $n \in \mathbb{Z}$, we can ask

- Is there is fixed point: $\varphi(x) = x$, or equivalently, $\varphi^n(x) = x$ for all n?
- Is there a periodic point: $\varphi^m(x) = x$ for some integer m?
- For a selected $x \in X$, what does the orbit of x looks like in X? The orbit of x is the collection of possible locations of x through all iterations of φ:
$$\mathrm{orb}(x) = \{\varphi^n(x) : n \in \mathbb{Z}\}.$$

First example

Let X be the real number line $\mathbb{R}$ and the map $\varphi : \mathbb{R} \to \mathbb{R}$ be given by $\varphi(t) = t + 1$, shifting the real number line to the right by 1:



In particular, for $t = 0$,

$$\ldots, \varphi^{-2}(0) = -2, \varphi^{-1}(0) = -1, \varphi^{0}(0) = 0, \varphi^{1}(0) = 1, \varphi^{2}(0) = 2, \ldots$$

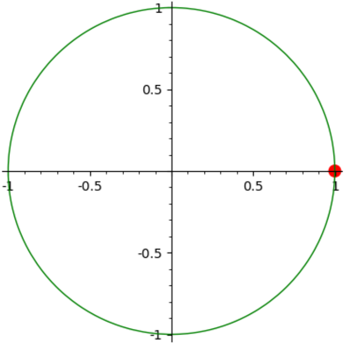
In other words, $\mathrm{orb}(0) = \mathbb{Z}$. Moreover, there is no fixed or periodic point.

Question: what if the map is $\varphi(t) = t^3$? (some stretching and compressing of the real number line with fixed points)

Rational rotation of the circle

Fix rational number $\theta = 1/4$. Let X be the unit circle in the complex plane: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\varphi(z) = e^{2\pi i\theta} z \in \mathbb{T}$.

Consider $z = e^{2\pi i} = 1 + 0i$,



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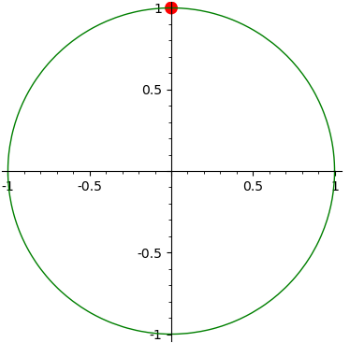
Consider $z = e^{2\pi i} = 1 + 0i$, $\varphi^1(z) = e^{2\pi i \frac{1}{4}} = 0 + 1i$



Rational rotation of the circle

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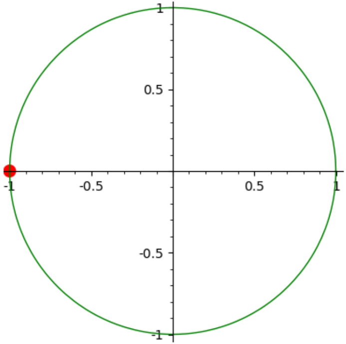
Consider $z = e^{2\pi i} = 1 + 0i$, $\varphi^2(z) = e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} = -1 + 0i$



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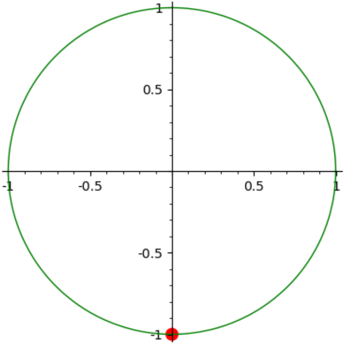
Consider $z = e^{2\pi i} = 1 + 0i$, $\varphi^3(z) = e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} = 0 - 1i$



Rational rotation of the circle

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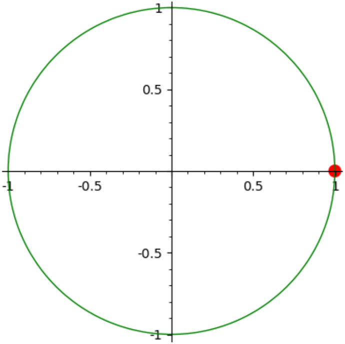
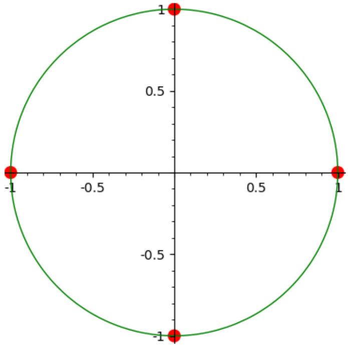
Consider $z = e^{2\pi i} = 1 + 0i$, $\varphi^4(z) = e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} e^{2\pi i \frac{1}{4}} = 1 + 0i = z$



Rational rotation of the circle

Together,

$$\text{orb}(z) = \{1 + 0i, 0 + 1i, -1 + 0i, 0 - 1i\}$$

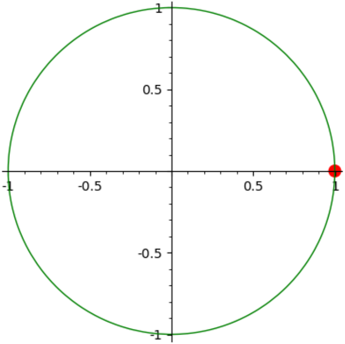


Question: what if we started with a different z?

Irrational rotation of the circle

Fix irrational number $\theta = \sqrt{2}$. Again, let X be the unit circle in the complex plane: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\varphi(z) = e^{2\pi i\theta} z$.

Consider $z = e^{2\pi i} = 1 + 0i$,



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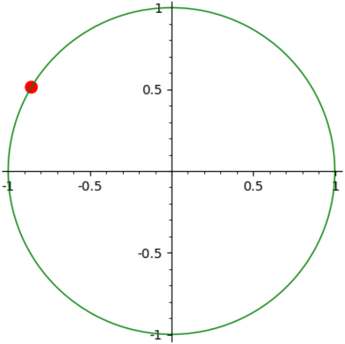
$\varphi(z) = e^{2\pi i \sqrt{2}}$,



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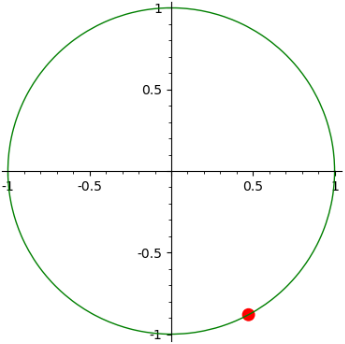
$\varphi^2(z) = e^{4\pi i\sqrt{2}}$,



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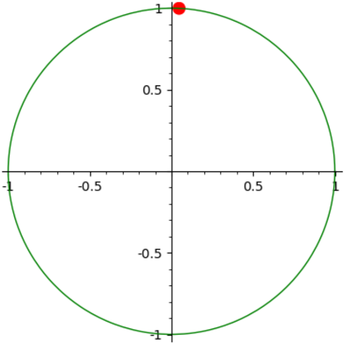
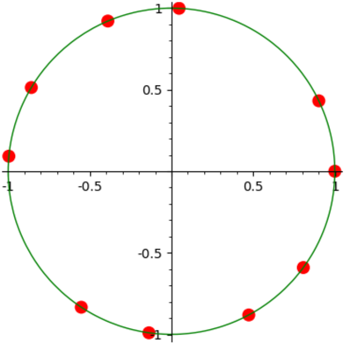
$\varphi^3(z) = e^{6\pi i\sqrt{2}}$,



Fix irrational number $\theta = \sqrt{2}$. Again, let X be the unit circle in the complex plane: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\varphi(z) = e^{2\pi i\theta} z$.

$\varphi^n(z)$ for $n = 0, 1, 2, \ldots, 9$ (they are 10 elements in $\mathrm{orb}(z)$)



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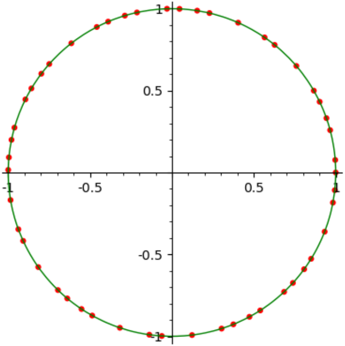
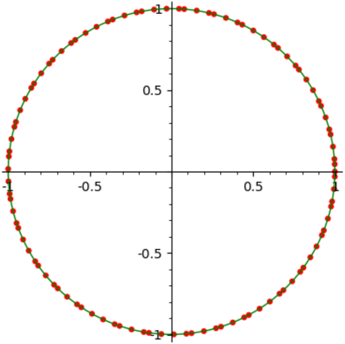
$\varphi^n(z)$ for $n = 0, 1, 2, \ldots, 49$ (they are 50 elements in orb(z))



Irrational rotation of the circle

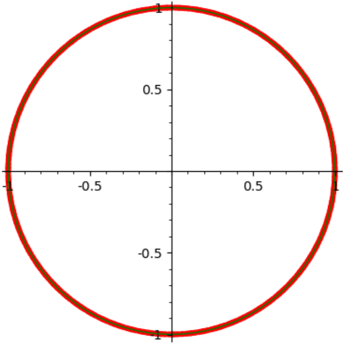
Fix irrational number $\theta = \sqrt{2}$. Again, let X be the unit circle in the complex plane: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\varphi(z) = e^{2\pi i\theta} z$.

$\varphi^n(z)$ for $n = 0, 1, 2, \ldots, 99$ (they are 100 elements in orb(z))

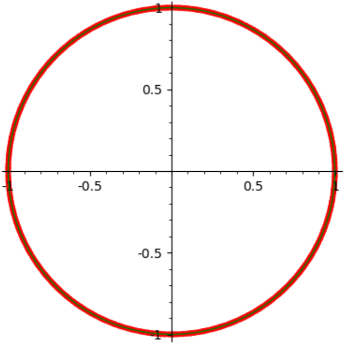


Fix irrational number $\theta = \sqrt{2}$. Again, let X be the unit circle in the complex plane: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\varphi(z) = e^{2\pi i\theta} z$.

$\varphi^n(z)$ for $n = 0, 1, 2, \ldots, 499$ (they are 500 elements in $\mathrm{orb}(z)$)



$\varphi^n(z)$ for all integers n (entire orbit of z)



In this situation, we say that orb(z) is dense in the circle $\mathbb{T}$.

Minimal dynamical systems

Question: what if we started with a different z?

Proposition

For θ an irrational number, $\mathrm{orb}(z)$ is dense in $\mathbb{T}$ for every $z \in \mathbb{T}$.

Intuitively, the Proposition suggests that the dynamical system from an irrational rotation of the circle is much more complex, and the dynamics involve "almost every" point on the circle $\mathbb{T}$.

Definition

A dynamical system (X, φ) is said to be minimal if for every point x in X, its orbit $\{\varphi^n(x) : n \in Z\}$ is dense in X.

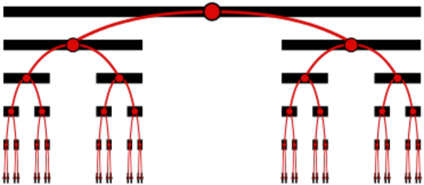
Odometer action

Consider the space $\mathcal{C}$ of infinite strings of 0's and 1's:

$$\mathcal{C} = \left\{ (a_i)_{i \in \mathbb{N}} : a_i \in \{0, 1\} \right\}$$

For example, $\cdots 1001011$, $\cdots 00000000$, $\cdots 11111111$ are all elements of $\mathcal{C}$.

With the appropriate metric on $\mathcal{C}$, it is a Cantor set (a non-empty, compact, totally disconnected metric space having no isolated points).

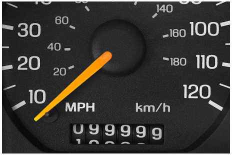


We write the sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathcal{C}$ from right to left: $\cdots a_3 a_2 a_1$ to make the map $\varphi : \mathcal{C} \to \mathcal{C}$ more intuitive: the map φ is given by "adding 1 and carry in base two". For example,

$$\begin{array}{r} \cdots 101\,011\,001\,011 \\ +\qquad\qquad\qquad\qquad 1 \\ \hline \cdots 101\,011\,001\,100 \end{array}$$

For the special element in $\mathcal{C}$: $\cdots 111111$ (the sequence of all 1's). We set $\varphi(\cdots 111111) = \cdots 000000$

Just like a car odometer, but infinite and performs addition in base two:



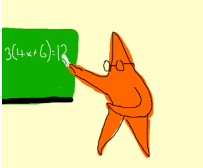
This is called the odometer action on the Cantor set. The dynamical system $(\mathcal{C}, \varphi)$ is also minimal.

A recurring theme in C*-algebra theory

Definition

A C*-algebra is a closed $*$-subalgebra of the bounded operators on a Hilbert space.

A C*-algebra is a rich algebraic structure with a "seemingly strict but very convenient" norm $\|\cdot\|$.



Commutative C*-algebras

Consider the space of all $\mathbb{C}$-valued continuous functions on the unit interval:

$$C([0,1]) := \{f : [0,1] \to \mathbb{C} : f \text{ is continuous}\}.$$

This space has many analytic and algebraic structures: for $f, g \in C([0,1])$,

- addition: $(f+g)(x) := f(x) + g(x)$
- multiplication $(f \cdot g)(x) := f(x) \cdot g(x)$
- scalar multiplication: $(\lambda f)(x) := \lambda f(x)$ (for λ in $\mathbb{C}$)
- involution: $f^*(x) := \overline{f(x)}$
- norm: $\|f\|_\infty = \sup_{x \in X} |f(x)|$
- ...
- C*-identity: $\|f^* \cdot f\|_\infty = \|f\|_\infty^2$

Example

The field of complex numbers $\mathbb{C}$ is a C*-algebra with involution given by complex conjugation $\lambda^* = \overline{\lambda}$.

Noncommutative C*-algebras

Fix n a strictly positive integer. Consider the collection of all $n \times n$ matrices with complex entries: $M_n(\mathbb{C})$. For $A, B \in M_n(\mathbb{C})$

- addition: $A + B$
- multiplication: AB
- scalar multiplication: λA (for λ in $\mathbb{C}$)
- involution: $A^*(x) := \overline{A^{\mathsf{T}}}$
- norm: $\|A\|_{\text{op}} = \inf \{c \geq 0 : \|Av\| \leq c \|v\| \text{ for all } v \in \mathbb{C}^n\}$
- ...
- C*-identity: $\|A^*A\|_{\text{op}} = \|A\|_{\text{op}}^2$

Example

The field of complex numbers $\mathbb{C}$ is a C*-algebra with involution given by complex conjugation $\lambda^* = \overline{\lambda}$.

Crossed product C*-algebra

Given a dynamical system, we can associate a C*-algebra that captures both the underlying space X and what the map φ does. Such a C*-algebra is called a crossed product C*-algebra and sometimes denoted by $C^*(X, \varphi)$.

Example

For θ irrational, $C^*(\mathbb{T}, \varphi)$ is called an irrational rotation algebra.

Example

$C^*(\mathcal{C}, \varphi)$ is an example of a Bunce-Deddens algebra.

Definition

A C*-algebra is simple if it does not contain any non-trivial closed two-sided ideals.

Loosely speaking, simple C*-algebras are the building blocks of C*-algebra theory. For example, $M_n(\mathbb{C})$ is simple for every $n \in \mathbb{N}$.

Theorem

Up to isomorphism of C*-algebra, every non-zero finite dimensional C*-algebra is of the form

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

for some integers $n_1, n_2, \cdots, n_k$.

Theorem

$C^*(X, \varphi)$ is simple if and only if (X, φ) is minimal.

Noncommutative solenoids

The noncommutative solenoids are a family of crossed product C*-algebras with underlying space the p-solenoid group $\mathcal{S}_p$ (inverse limit of circles connected by p-fold self-covers). We form the C*-algebra by letting the group $\mathbb{Z}\left[\frac{1}{p}\right]$ act on $\mathcal{S}_p$ by rotation of the circle at each stage. Alternatively, a noncommutative solenoid is an inverse limit of rotation algebras.

A p-solenoid $\mathcal{S}_p$ is a compact metrizable space that is connected, but not locally connected or path connected. The group $\mathbb{Z}\left[\frac{1}{p}\right]$ is a discrete group that is not finitely generated.

Question: what is a necessary and sufficient condition for two noncommutative solenoids to be (strongly) Morita equivalent?

References

1. K. R. Davidson, C*-algebras by examples
2. G. Murphy, C*-algebras and operator theory
3. I. Putnam, Cantor Minimal Systems
4. D. P. Williams, Crossed products of C*-algebras

Thank you!