Constructing Finitely Generated Projective Modules for Noncommutative Solenoids

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Interest and Motivation

- Twisted group C*-algebras associated to non-finitely generated groups
- Noncommutative torus is an important example in noncommutative geometry and has been intensely studied
- Heisenberg bimodules provide the natural setting for studying duality theory of Gabor systems
- Solenoid group, *p*-adic analysis, etc.

Twisted Group C*-Algebra

Let Γ be a discrete locally compact group. A *multiplier* σ on Γ is a normalized group 2-cocycle on Γ with values in $\mathbb{T}$ (for the trivial group action of Γ on $\mathbb{T}$): for all $r, s, t \in \Gamma$, identity $e \in \Gamma$,

1. $\sigma(r, s)\sigma(rs, t) = \sigma(r, st)\sigma(s, t)$;
2. $\sigma(s, e) = \sigma(e, s) = 1$.

The twisted group C*-algebra $C^*(\Gamma, \sigma)$ is the C*-enveloping algebra of $\ell^1(\Gamma, \sigma)$, which is $\ell^1(\Gamma)$ with *twisted convolution and involution*: for all $s \in \Gamma$,

$$f * g(s) = \sum_{t \in \Gamma} f(t)g(t^{-1}s)\sigma(t, t^{-1}s);$$

$$f^*(s) = \overline{\sigma(s, -s)f(-s)}.$$

Fix prime p. Define

$$\mathbb{Z}\left[\frac{1}{p}\right] := \left\{\frac{j}{p^k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N}\right\},$$

and denote by Γ the discrete abelian group $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$. We wish to study $C^*(\Gamma, \sigma)$.

Theorem (Packer, Latrémolière, '13)

For a fixed prime p, the set
$\Xi_p := \{(\alpha_n)_{n\in\mathbb{N}} : \alpha_0 \in [0,1) \text{ and } \forall n \in \mathbb{N}, \exists b_n \in \{0,\ldots,p-1\} \text{ such that } p\alpha_{n+1} = \alpha_n + b_n\}$
forms a group under pointwise addition modules one. Every multiplier on Γ is cohomologous to the multiplier:

$$\Psi_\alpha : \begin{cases} \Gamma \times \Gamma & \to & \mathbb{T} \\ \left(\left(\frac{j_1}{p^{k_1}}, \frac{j_2}{p^{k_2}}\right), \left(\frac{j_3}{p^{k_3}}, \frac{j_4}{p^{k_4}}\right)\right) & \mapsto & \exp\left(2\pi i \alpha_{(k_1+k_4)} j_1 j_4\right). \end{cases}$$

Additionally, Ψ_α and Ψ_β are cohomologous iff $\alpha = \beta$.

$$\Xi_p := \{(\alpha_n)_{n\in\mathbb{N}} : \alpha_0 \in [0,1) \text{ and } \forall n \in \mathbb{N}, \exists b_n \in \{0,\ldots,p-1\} \text{ such that } p\alpha_{n+1} = \alpha_n + b_n\}$$

Note:

1. $\Xi_p \cong \mathscr{S}_p$, where $\mathscr{S}_p$ is the p-solenoid group.
2. For $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \Xi_p$, if α_N is irrational for any N, then α is a sequence of distinct irrational numbers.
3. We can associate a unique p-adic integer $x_\alpha = \sum_{j=0}^{\infty} b_n p^n$ for each $\alpha \in \Xi_p$.

Definition

For a fixed prime p and $\alpha \in \Xi_p$, we denote by $\mathscr{A}_\alpha^{\mathscr{S}}$ the twisted group C*-algebra $C^*(\Gamma, \Psi_\alpha)$, and call them *noncommutative solenoids*.

Alternative Definition 1. Let A_θ denote the rotation C*-algebra, then

$$A_{\alpha_0} \xrightarrow{\varphi_0} A_{\alpha_2} \xrightarrow{\varphi_1} A_{\alpha_4} \xrightarrow{\varphi_2} \cdots$$

converges to $\mathscr{A}_\alpha^{\mathscr{S}}$.

Alternative Definition 2. Let ρ^α be the action of $\mathbb{Z}[1/p]$ on $\mathscr{S}_p$ defined by

$$\rho^\alpha_{\frac{j}{p^k}}\left((z_n)_{n\in\mathbb{N}}\right) = \left(\exp\left(2\pi i \alpha_{k+n} j\right) z_n\right)_{n\in\mathbb{N}}.$$

Then $C\left(\mathscr{S}_p\right) \rtimes_{\rho^\alpha} \mathbb{Z}[1/p]$ is $*$-isomorphic to $\mathscr{A}_\alpha^{\mathscr{S}}$.

Strong Morita Equivalence

Two C*-algebras A and B are *(strongly) Morita equivalent* if there exists an A-B-equivalence bimodule.

Definition

A Banach A-B-bimodule X is called an *A-B-equivalence bimodule* if it is both a left Hilbert A-module and a right Hilbert B-module such that

- The ideals ${}_A\langle X, X\rangle$ and $\langle X, X\rangle_B$ are dense in A and B, respectively.
- $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ and ${}_A\langle x \cdot b, y\rangle =_A \langle x, y \cdot b^*\rangle$
- ${}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B$.

Question

Given α, $\beta \in \Xi_p$, is $\mathscr{A}_\alpha^{\mathscr{S}}$ Morita equivalent to $\mathscr{A}_\beta^{\mathscr{S}}$?

Question

Given $\alpha \in \Xi_p$, what are some $\beta \in \Xi_p$ such that $\mathscr{A}_\alpha^{\mathscr{S}}$ Morita equivalent to $\mathscr{A}_\beta^{\mathscr{S}}$?

Heisenberg Equivalence Bimodules

We follow a construction of Rieffel. Let M be a locally compact abelian group, and $G := M \times \widehat{M}$. The *Heisenberg multiplier* $\eta : (M \times \widehat{M}) \times (M \times \widehat{M})$ is given by

$$\eta\left((m, s), (n, t)\right) = \langle m, t \rangle.$$

Let D be a lattice in $M \times \widehat{M}$, and denote by $D^{\perp}$ the lattice in $M \times \widehat{M}$ given by

$$D^{\perp} = \left\{ (n, t) \in M \times \widehat{M} : \forall (m, s) \in D, \eta\left((m, s), (n, t)\right) \overline{\eta\left((n, t), (m, s)\right)} = 1 \right\}$$

Theorem (Rieffel '88)

There is a $C^*(D, \eta)$-$C^*(D^{\perp}, \overline{\eta})$-equivalence bimodule.

Embed $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$ into $M \times \widehat{M}$

Let $M := \mathbb{Q}_p \times \mathbb{R}$, where $\mathbb{Q}_p$ denote the field of p-adic numbers. Recall that for a fix prime p, a p-adic number is a formal series

$$\sum_{j=v}^{\infty} a_j p^j, \quad a_j \in \{0, 1, \ldots, p-1\}, \quad v \in \mathbb{Z},$$

and $\mathbb{Q}_p$ is self-duel with the pairing $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{T}$, $\langle x, y \rangle = e^{2\pi i \{xy\}_p}$, where $\{x\}_p$ denotes the fractional part of x:

$$\left\{ \sum_{j=v}^{\infty} a_j p^j \right\}_p = \sum_{j=v}^{-1} a_j p^j.$$

Then M is self-dual.

Embed $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$ into $M \times \widehat{M}$

We have $M = \widehat{M} = \mathbb{Q}_p \times \mathbb{R}$. For any $(x, \theta) \in [\mathbb{Q}_p \setminus \{0\}] \times [\mathbb{R} \setminus \{0\}]$, the map

$$\iota_{x,\theta} : \begin{cases} \mathbb{Z}[1/p] \times \mathbb{Z}[1/p] & \rightarrow \quad [\mathbb{Q}_p \times \mathbb{R}] \times [\mathbb{Q}_p \times \mathbb{R}] \\ (r_1 \times r_2) & \mapsto [(x \cdot r_1, \theta \cdot r_1), (r_2, r_2)] . \end{cases}$$

is an embedding, and we denote the image of $\iota_{x,\theta}$ by $D_{x,\theta}$. One can check that $D_{x,\theta}^{\perp} \cong \mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$.

Theorem (Packer, Latrémolière, '13)

For every nonzero $\alpha \in \Xi_p$ with $b_n \neq 0$ for some $n \in \mathbb{N}$, $C^*(D_{x,\theta}, \eta)$ is $*$-isomorphic to $\mathscr{A}_\alpha^{\mathscr{S}}$ for $x = x_\alpha$ and $\theta = \alpha_0$. Furthermore, $C^*(D_{x,\theta}^{\perp}, \overline{\eta})$ is $*$-isomorphic to a noncommutative solenoid.

Recall that for each $\alpha \in \Xi_p$ with $\alpha_{n+1} = p\alpha_n + b_n$ for all $n \in \mathbb{N}$, we can associate a unique p-adic number $x_\alpha = \sum_{j=0}^{\infty} b_j p^j$.

Proposition (L.)

Same set up as the theorem above, and let $x_\alpha^{-1} = \sum_{j=v}^{\infty} a_j p^j \in \mathbb{Q}_p$. Then $C^*(D^{\perp}_{x,\theta}, \overline{\eta})$ is $*$-isomorphic to $\mathscr{A}_\beta^{\mathscr{S}}$, where $\beta = (\beta_n)_{n\in\mathbb{N}}$ is given by

$$\beta_n = \frac{1}{\theta p^n} + \frac{\sum_{j=v}^{n-1} a_j p^j}{p^n}.$$

Compare with $x_\alpha = \sum_{j=v}^{\infty} b_j p^j$, and

$$\alpha_n = \frac{\alpha_0 + \sum_{j=0}^{n-1} b_j p^j}{p^n}.$$

This establishes Morita equivalence between $\mathscr{A}_\alpha^{\mathscr{S}}$ and $\mathscr{A}_\beta^{\mathscr{S}}$.

Building projective bimodules from the "inside-out"

Let A_θ denote the rotation C*-algebra. Recall that $\mathscr{A}_\alpha^{\mathscr{S}} = \varinjlim A_{\alpha_{2n}}$. We wish to establish equivalence bimodule X_{2n} between $A_{\alpha_{2n}}$ and $A_{\beta_{2n}}$:

$$
\begin{array}{ccccccc}
A_{\alpha_0} & \xrightarrow{\varphi_0} & A_{\alpha_2} & \xrightarrow{\varphi_1} & A_{\alpha_4} & \xrightarrow{\varphi_2} & \cdots \\
\Big| & & \Big| & & \Big| & & \\
X_0 & \xrightarrow{i_0} & X_2 & \xrightarrow{i_1} & X_4 & \xrightarrow{i_2} & \cdots \\
\Big| & & \Big| & & \Big| & & \\
A_{\beta_0} & \xrightarrow{\psi_0} & A_{\beta_2} & \xrightarrow{\psi_1} & A_{\beta_4} & \xrightarrow{\psi_2} & \cdots
\end{array}
$$

Lemma (Rieffel '82)

Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let P be a nontrivial projection in A_θ such that $\tau(P) = c\theta + d$ with c and d generating $\mathbb{Z}$. Set $\lambda = \frac{a\theta+b}{c\theta+d}$ for any $a, b \in \mathbb{Z}$ such that $ad - bc = \pm 1$. Then $PA_\theta P \cong A_\lambda$.

In the case when $\alpha \in \Xi_p$ is an irrational sequence,

$$\begin{array}{ccccccc}
A_{\alpha_0} & \xrightarrow{\varphi_0} & A_{\alpha_2} & \xrightarrow{\varphi_1} & A_{\alpha_4} & \xrightarrow{\varphi_2} & \cdots \\
| & & | & & | & & \\
X_0 & \xrightarrow{i_0} & X_2 & \xrightarrow{i_1} & X_4 & \xrightarrow{i_2} & \cdots \\
| & & | & & | & & \\
PA_{\alpha_0}P & \xrightarrow{\psi_0} & PA_{\alpha_2}P & \xrightarrow{\psi_1} & PA_{\alpha_4}P & \xrightarrow{\psi_2} & \cdots \\
\| & & \| & & \| & & \\
A_{\beta_0} & \xrightarrow{\psi_0} & A_{\beta_2} & \xrightarrow{\psi_1} & A_{\beta_4} & \xrightarrow{\psi_2} & \cdots
\end{array}$$

Under certain conditions on the projection P, $\varinjlim PA_{\alpha_n}P$ converges to some noncommutative solenoid $\mathscr{A}_\beta^{\mathscr{S}}$, establishing Morita equivalence between $\mathscr{A}_\alpha^{\mathscr{S}}$ and $\mathscr{A}_\beta^{\mathscr{S}}$.

Theorem (L.)

Let p be prime and $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in \Xi_p$ an irrational sequence such that $x_\alpha \in \mathbb{Z}_p^\times$. Then the Heisenberg bimodule construction gives the same Morita equivalence bimodules as building equivalence bimodules from the "inside out" using a fixed projection $P \in A_{\alpha_0}$ with $\tau_0(P) = \alpha_0$.

Details: $\tau_{2n}(P) = p^{2n}\alpha_{2n} - \sum_{j=0}^{2n-1} b_j p^j$, that is,

$$c_{2n} = p^{2n} \text{ and } d_{2n} = -\sum_{j=0}^{2n-1} b_j p^j.$$

Taking

$$a_{2n} = \sum_{j=0}^{2n-1} a_j p^j \text{ and } b_{2n} = p^{-2n}\left(\left(\sum_{j=0}^{2n-1} a_j p^j\right)\left(-\sum_{j=0}^{2n-1} b_j p^j\right) + 1\right),$$

and setting $\beta_{2n} = (a_{2n}\alpha_{2n} + b_{2n})/(c_{2n}\alpha_{2n} + d_{2n})$, we obtain $\beta = (\beta_n)_{n\in\mathbb{N}}$, which turns out to be the same β as the one from the Heisenberg module.

Example. Fix $p = 3$ prime and an irrational sequence $\alpha \in \Xi_3$ by

$$(\alpha_n)_{n\in\mathbb{N}} = \left(\theta, \frac{\theta+2}{3}, \frac{\theta+5}{9}, \frac{\theta+23}{27}, \frac{\theta+50}{81}, \frac{\theta+212}{243}, \frac{\theta+455}{729}, \dots\right),$$

which determines the p-adic integer $x = \sum_{j=0}^{\infty} b_j p^j$ with

$$(b_j)_{j\in\mathbb{N}} = (2,1,2,1,2,1,\dots) = (\overline{2,1}).$$

Then $x^{-1} = \sum_{j=0}^{\infty} a_j p^j \in \mathbb{Z}_3$, where

$$(a_j)_{j\in\mathbb{N}} = (2,0,2,1,0,1,2,1,0,1,\dots) = (2,0,\overline{2,1,0,1}).$$

Via the Heisenberg bimodules construction, we get

$$(\beta_n)_{n\in\mathbb{N}} = \left(\frac{1}{\theta}, \frac{2\theta+1}{3\theta}, \frac{2\theta+1}{9\theta}, \frac{20\theta+1}{27\theta}, \frac{47\theta+1}{81\theta}, \frac{47\theta+1}{243\theta}, \frac{290\theta+1}{729\theta}, \dots\right).$$

It is easy to check that A_{α_j} is Morita equivalent to A_{β_j} for $j = 0, 2, 4$ and 6.

References

1. F. Latrémolière and J. Packer, *Noncommutative solenoids*, New York J. Math., 2018
2. F. Latrémolière and J. Packer, *Explicit construction of equivalence bimodules between noncommutative solenoids*, Vol. 650. Contemp. Math. Amer. Math. Soc., 2013
3. M. Rieffel, *The cancellation theorem for projective modules over irrational rotation C*-algebras*, Proc. London Math., 1983
4. M. Rieffel, *Projective modules over higher-dimensional non-commutative tori*, Canad. J. Math., 1988
5. A. M. Robert, *A course in p-adic analysis*, Vol. 198, Graduate Texts in Mathematics. Springer, 2000

Thank you!