Irrational noncommutative solenoids and their finitely generated projective modules

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Outline

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   - Direct limit of rotation algebras
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3. Directed systems of equivalence bimodules (building bimodules from the "inside-out")
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Interest and Motivation

- Twisted group C*-algebras associated to non-finitely generated groups:

$$\mathbb{Z}\left[\frac{1}{p}\right] := \left\{\frac{j}{p^k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N}\right\}$$

- Rotation algebra, noncommutative geometry
- Other applications of Heisenberg bimodules and directed systems of equivalence bimodules
- Relation to p-adic analysis, etc.

Noncommutative Solenoid

Fix prime p. Define

$$\mathbb{Z}\left[\frac{1}{p}\right] := \left\{\frac{j}{p^k} \in \mathbb{Q} : z \in \mathbb{Z}, k \in \mathbb{N}\right\},$$

and denote by Γ the discrete abelian group $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$. We wish to study $C^*(\Gamma, \sigma)$, where $\sigma : \Gamma \times \Gamma \to \mathbb{T}$ is a *multiplier* (normalized group 2-cocycle on Γ with values in $\mathbb{T}$).

Theorem (Packer, Latrémolière, '13)

For a fixed prime p, the set
$\Xi_p := \{(\alpha_n)_{n\in\mathbb{N}} : \alpha_0 \in [0,1) \text{ and } \forall n \in \mathbb{N}, \exists x_n \in \{0,\ldots,p-1\} \text{ such that } p\alpha_{n+1} = \alpha_n + x_n\}$
forms a group under pointwise addition modulus one. Every multiplier on Γ is cohomologous to the multiplier:

$$\Psi_\alpha : \begin{cases} \Gamma \times \Gamma & \to \quad \mathbb{T} \\ \left(\left(\frac{j_1}{p^{k_1}}, \frac{j_2}{p^{k_2}}\right), \left(\frac{j_3}{p^{k_3}}, \frac{j_4}{p^{k_4}}\right)\right) & \mapsto \exp\left(2\pi i \alpha_{(k_1+k_4)} j_1 j_4\right). \end{cases}$$

Additionally, Ψ_α and Ψ_β are cohomologous iff $\alpha = \beta$.

Definition

For a fixed prime p and $\alpha \in \Xi_p$, we denote by $\mathscr{A}_\alpha^{\mathscr{S}}$ the twisted group C*-algebra $C^*(\Gamma, \Psi_\alpha)$, and call them *noncommutative solenoids*.

$$\Xi_p := \{(\alpha_n)_{n\in\mathbb{N}} : \alpha_0 \in [0,1) \text{ and } \forall n \in \mathbb{N}, \exists x_n \in \{0, \ldots, p-1\} \text{ such that } p\alpha_{n+1} = \alpha_n + x_n\}$$

- $\Xi_p \cong \mathscr{S}_p$, where $\mathscr{S}_p$ is the p-solenoid group.
- We can associate a unique p-adic integer $x_\alpha = \sum_{j=0}^{\infty} x_n p^n$ for each $\alpha \in \Xi_p$. (For a fix prime p, a p-adic number is a formal series $\sum_{j=v}^{\infty} x_j p^j$ with $x_j \in \{0, 1, \ldots, p-1\}$ and $v \in \mathbb{Z}$.

Lemma

Let

$$\Omega_p := \{(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \forall n \in \mathbb{N}, \exists x_n \in \mathbb{Z} \text{ such that } p\alpha_{n+1} = \alpha_n + x_n\},$$

which is a group under pointwise addition. The map $h : \Omega_p \to \Xi_p$ given by $h((a_n)_{n\in\mathbb{N}}) = (a_n \mod \mathbb{Z})_{n\in\mathbb{N}}$ defines a surjective group homomorphism. For $\alpha' \in \Omega_p$ and $\alpha \in \Xi_p$, $\mathscr{A}_{\alpha'}^{\mathscr{S}} \cong \mathscr{A}_\alpha^{\mathscr{S}}$ if and only if $h(\alpha') = \alpha$.

Alternative Definitions

Alternative Definition 1. Let A_θ denote the rotation algebra associated to θ, then

$$A_{\alpha_0} \xrightarrow{\varphi_0} A_{\alpha_2} \xrightarrow{\varphi_1} A_{\alpha_4} \xrightarrow{\varphi_2} \cdots$$

converges to $\mathscr{A}_\alpha^{\mathscr{S}}$, where $\varphi_n : A_{\alpha_{2n}} \to A_{\alpha_{2n+2}}$ is the unique homomorphism given by

$$\varphi_n(U_{\alpha_{2n}}) = U^p_{\alpha_{2n+2}} \quad \text{and} \quad \varphi_n(V_{\alpha_{2n}}) = V^p_{\alpha_{2n+2}}.$$

Alternative Definition 2. Let ρ^α be the action of $\mathbb{Z}[1/p]$ on $\mathscr{S}_p$ defined by

$$\rho^\alpha_{\frac{j}{p^k}}\left((z_n)_{n\in\mathbb{N}}\right) = \left(\exp\left(2\pi i \alpha_{k+n} j\right) z_n\right)_{n\in\mathbb{N}}.$$

Then $C(\mathscr{S}_p) \rtimes_{\rho^\alpha} \mathbb{Z}[1/p]$ is $*$-isomorphic to $\mathscr{A}_\alpha^{\mathscr{S}}$.

There are three distinct subclasses of NC solenoids based on $\alpha \in \Xi_p$:

- α is periodic (necessarily rational): *periodic rational nc solenoid*
- α is rational and aperiodic: *aperiodic rational nc solenoid*
- α is irrational (necessarily aperiodic), *irrational nc solenoid*

Theorem (Packer, Latrémolière, '13)

$K_0\left(\mathscr{A}_\alpha^{\mathscr{S}}\right) =$ an Abelian extension of $\mathbb{Z}\left[\frac{1}{p}\right]$ by $\mathbb{Z}$ determined by x_α

$K_1\left(\mathscr{A}_\alpha^{\mathscr{S}}\right) = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$

More interestingly, all tracial states of of $\mathscr{A}_\alpha^{\mathscr{S}}$ lift to a single trace τ on $K_0\left(\mathscr{A}_\alpha^{\mathscr{S}}\right)$ with range

$$\tau\left(K_0\left(\mathscr{A}_\alpha^{\mathscr{S}}\right)\right) = \varinjlim \mathbb{Z} \oplus \alpha_n\mathbb{Z} = \{z + y\alpha_n \ : z, y \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Strong Morita Equivalence

Two C*-algebras A and B are *(strongly) Morita equivalent* if there exists an A-B-equivalence bimodule.

Definition

An A-B-bimodule X is called an *A-B-equivalence bimodule* if it is both a left Hilbert A-module and a right Hilbert B-module such that

- The ideals ${}_A\langle X, X\rangle$ and $\langle X, X\rangle_B$ are dense in A and B, respectively.
- $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ and ${}_A\langle x \cdot b, y\rangle = {}_A\langle x, y \cdot b^*\rangle$
- ${}_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B$.

Many algebraic properties are preserved under Morita equivalence: ideal structure, representation theory, K-theory, etc.

Example

For any C*-algebra A and a full projection $P \in M_k(A)$ (i.e. $\overline{M_k(A)PM_k(A)} = M_k(A)$), PA^k is a $PM_k(A)P$-A-equivalence bimodule with

$$\langle Pa, Pb \rangle_A = \sum_{i=1}^{k} u_i^* v_i \quad \text{and} \quad \left(\langle Pa, Pb \rangle_{PM_k(A)P} \right)_{ij} = u_i v_j^*,$$

for $Pa = [u_1, \ldots, u_k], Pb = [v_1, \ldots, v_k] \in PA^k$.

Theorem (Rieffel, '81)

If A and B are unital C*-algebras that are Morita equivalent, then each is a full corner of the algebra of $n \times n$ matrices over the other.

Question

Given α, $\beta \in \Xi_p$, is $\mathscr{A}_\alpha^{\mathscr{S}}$ Morita equivalent to $\mathscr{A}_\beta^{\mathscr{S}}$?

Approach:

- If $\alpha \in \Xi_p$, $\beta \in \Xi_q$, and $p \neq q$, then $K_1\left(\mathscr{A}_\alpha^{\mathscr{S}}\right) \neq K_1\left(\mathscr{A}_\beta^{\mathscr{S}}\right)$
- There must be some $m \in \mathbb{N}$ and projection $P \in M_m\left(\mathscr{A}_\alpha^{\mathscr{S}}\right)$ such that $\mathscr{A}_\beta^{\mathscr{S}} \cong PM_m\left(\mathscr{A}_\alpha^{\mathscr{S}}\right)P$.
- Any $P \in M_m\left(\mathscr{A}_\alpha^{\mathscr{S}}\right)$ is unitarily equivalent to $P \in M_m\left(A_{\alpha_k}\right)$ for some $k \in \mathbb{N}$.
- What are some necessary and sufficient conditions we can put on this P so that we can "rebuild" $\mathscr{A}_\beta^{\mathscr{S}}$ by considering $\varinjlim_{n\geq k} PM_m\left(A_{\alpha_{2n}}\right)P$?

Directed systems of equivalence bimodules

Consider two directed systems of unital C*-algebras, whose *-morphisms are all unital maps:

$$A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$$

and

$$B_0 \xrightarrow{\psi_0} B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} \cdots .$$

A sequence $(X_n, i_n)_{n\in\mathbb{N}}$ is a directed system of equivalence bimodule adapted to the sequence $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ when X_n is an A_n-B_n-equivalence bimodule such that the sequence

$$X_0 \xrightarrow{\iota_0} X_1 \xrightarrow{\iota_1} X_2 \xrightarrow{\iota_2} \cdots$$

is a directed sequence of modules satisfying

$$\langle \iota_n(f), \iota_n(g)\rangle_{B_{n+1}} = \psi_n\left(\langle f, g\rangle_{B_n}\right), \quad \text{for all } f, g \in X_n,$$

and

$$\iota_n(f \cdot b) = \iota_n(f) \cdot \psi_n(b), \quad \text{for all } f \in X_n, b \in B_n,$$

with analogous but symmetric equalities holding for the X_n viewed as left Hilbert A_n-modules:

Building projective bimodules from the "inside-out"

Recall that $\mathscr{A}_\alpha^{\mathscr{S}} = \varinjlim A_{\alpha_{2n}}$. We wish to establish equivalence bimodule X_{2n} between $A_{\alpha_{2n}}$ and $A_{\beta_{2n}}$:

$$
\begin{array}{ccccccc}
A_{\alpha_0} & \xrightarrow{\varphi_0} & A_{\alpha_2} & \xrightarrow{\varphi_1} & A_{\alpha_4} & \xrightarrow{\varphi_2} & \cdots \\
\Big| & & \Big| & & \Big| & & \\
X_0 & \xrightarrow{\iota_0} & X_2 & \xrightarrow{\iota_1} & X_4 & \xrightarrow{\iota_2} & \cdots \\
\Big| & & \Big| & & \Big| & & \\
A_{\beta_0} & \xrightarrow{\psi_0} & A_{\beta_2} & \xrightarrow{\psi_1} & A_{\beta_4} & \xrightarrow{\psi_2} & \cdots
\end{array}
$$

In the case when $\alpha \in \Xi_p$ is an irrational sequence, the idea is to find a projection $P \in M_k\left(A_{\alpha_0}\right) \subset M_k\left(A_{\alpha_{2n}}\right)$ for all n, and consider $PM_k\left(A_{\alpha_{2n}}\right)P$:

$$\begin{tikzcd}
A_{\alpha_0} \arrow[r, "\varphi_0"] \arrow[d, dash] & A_{\alpha_2} \arrow[r, "\varphi_1"] \arrow[d, dash] & A_{\alpha_4} \arrow[r, "\varphi_2"] \arrow[d, dash] & \cdots \\
X_0 \arrow[r, "\iota_0"] \arrow[d, dash] & X_2 \arrow[r, "\iota_1"] \arrow[d, dash] & X_4 \arrow[r, "\iota_2"] \arrow[d, dash] & \cdots \\
PM_k\left(A_{\alpha_0}\right)P \arrow[r, "\psi_0"] \arrow[d, equal] & PM_k\left(A_{\alpha_2}\right)P \arrow[r, "\psi_1"] \arrow[d, equal] & PM_k\left(A_{\alpha_4}\right)P \arrow[r, "\psi_2"] \arrow[d, equal] & \cdots \\
A_{\beta_0} \arrow[r, "\psi_0"] & A_{\beta_2} \arrow[r, "\psi_1"] & A_{\beta_4} \arrow[r, "\psi_2"] & \cdots
\end{tikzcd}$$

Under certain conditions on the projection P, $\varinjlim PM_k\left(A_{\alpha_n}\right)P$ converges to a noncommutative solenoid $\mathscr{A}_{\beta}^{\mathscr{S}}$.

Rieffel's standard bimodule

Fix α a nonzero real number. Let c and d be a pair of integers that generate $\mathbb{Z}$, such that $c\alpha + d \neq 0$ and $c \neq 0$. Let $\gamma = (c\alpha + d)^{-1}$. For $\beta = (a\alpha + b)\gamma$, where a and b are integers such that $ad - bc = \pm 1$.

Theorem (Rieffel, '83)

Let $G = \mathbb{R} \times \mathbb{Z}_{|c|}$ and consider the following subgroups of G:

$$H = \{(n, [dn]_c) : n \in \mathbb{Z}\}, \qquad K = \{(n\gamma, [n]_c) : n \in \mathbb{Z}\}.$$

Let H act on $K\backslash G$ (the right cosets of K) by right translation, and let K act on G/H (the left cosets of H) by left translation. Then the transformation group C*-algebras $C^*(H, K\backslash G)$ and $C^*(K, G/H)$ are isomorphic to A_α and A_β, respectively. Furthermore, $C_c(G)$, suitably completed and structured, provides an A_β-A_α-equivalence bimodule.

Lemma (Kodaka '92)

Let P be a projection in $M_k(A_\alpha)$ with (unnormalized) trace $c\alpha + d$ (with the same conditions), then Rieffel's standard bimodules is isomorphic to PA_α^k. Therefore, $PM_k(A_\alpha)P \cong A_\beta$.

Let P be a nontrivial projection in $M_k(A_{\alpha_0})$ with (unnormalized) trace $c_0\alpha + d_0$ such that (c_0, d_0) generate $\mathbb{Z}$.
As a projection in $M_k(A_{\alpha_{2n}})$,

$$\tau(P) = c_0\alpha_0 + d_0 = \left(c_0 p^{2n}\right)\alpha_{2n} + \left(d_0 - c_0 \sum_{j=0}^{2n-1} x_j p^j\right) = c_{2n}\alpha_{2n} + d_{2n}$$

Key Condition

$\gcd(c_0 p, d_0 - c_0 x_0) = 1.$

Lemma

If $\gcd(c_0 p, d_0 - c_0 x_0) = 1$, then $\gcd(c_{2n}, d_{2n}) = 1$ for all $n \in \mathbb{N}$.

Punchline

Theorem (L.)

Let P be a projection in $M_k(A_{\alpha_0})$ that satisfies the Key Condition. Then there exists $\beta \in \Xi_p$ such that $\mathscr{A}_\alpha^{\mathscr{S}}$ is Morita equivalent to $\mathscr{A}_\beta^{\mathscr{S}} = \varinjlim A_{\beta_{2n}}$, where for each n

$$\beta_{2n} = (a_{2n}\alpha_{2n} + b_{2n}) / (c_{2n}\alpha_{2n} + d_{2n}) \in [0,1)$$

for a unique pair of integers a_{2n} and b_{2n} such that $a_{2n}d_{2n} - b_{2n}c_{2n} = \pm 1$. This β is uniquely determined by $\alpha \in \Xi_p$ and the projection P.

The Key Condition is also necessary in the following sense:

Corollary

Fix prime p. Let $\alpha, \beta \in \Xi_p$. Then the following statements are equivalent.

1. $\mathscr{A}_\alpha^{\mathscr{S}}$ and $\mathscr{A}_\beta^{\mathscr{S}}$ are Morita equivalent.
2. There exist $k, N \in \mathbb{N}$ and a projection P satisfying the Theorem above for $(\beta_n)_{n \geq N}$.

$$\begin{tikzcd}
A_{\alpha_0} \arrow[r, "\varphi_0"] \arrow[d, dash] & A_{\alpha_2} \arrow[r, "\varphi_1"] \arrow[d, dash] & A_{\alpha_4} \arrow[r, "\varphi_2"] \arrow[d, dash] & \cdots \\
X_0 \arrow[r, "\iota_0"] \arrow[d, dash] & X_2 \arrow[r, "\iota_1"] \arrow[d, dash] & X_4 \arrow[r, "\iota_2"] \arrow[d, dash] & \cdots \\
PM_k\left(A_{\alpha_0}\right)P \arrow[r, "\psi_0"] \arrow[d, equal] & PM_k\left(A_{\alpha_2}\right)P \arrow[r, "\psi_1"] \arrow[d, equal] & PM_k\left(A_{\alpha_4}\right)P \arrow[r, "\psi_2"] \arrow[d, equal] & \cdots \\
A_{\beta_0} \arrow[r, "\psi_0"] & A_{\beta_2} \arrow[r, "\psi_1"] & A_{\beta_4} \arrow[r, "\psi_2"] & \cdots
\end{tikzcd}$$

Relating the two constructions

Example

Let p be prime and $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in \Xi_p$ an irrational sequence such that $x_0 \neq 0$. Then the projection $P \in A_{\alpha_0}$ with $\tau_0(P) = \alpha_0$ satisfies the Key Condition and uniquely determines a $\beta \in \Xi_p$ such that $\mathscr{A}^{\mathscr{S}}_{\alpha} \sim_M \mathscr{A}^{\mathscr{S}}_{\beta}$.)

Details: $\tau_{2n}(P) = p^{2n}\alpha_{2n} - \sum_{j=0}^{2n-1} b_j p^j$, that is,

$$c_{2n} = p^{2n} \text{ and } d_{2n} = -\sum_{j=0}^{2n-1} x_j p^j.$$

Taking

$$a_{2n} = \sum_{j=0}^{2n-1} y_j p^j \text{ and } b_{2n} = p^{-2n}\left(\left(\sum_{j=0}^{2n-1} y_j p^j\right)\left(-\sum_{j=0}^{2n-1} x_j p^j\right) + 1\right),$$

and setting $\beta_{2n} = (a_{2n}\alpha_{2n} + b_{2n})/(c_{2n}\alpha_{2n} + d_{2n})$, we obtain $\beta = (\beta_n)_{n\in\mathbb{N}}$, which will turn out to be the same β as the one from the Heisenberg module.

Heisenberg Equivalence Bimodules

We follow a construction of Rieffel. Let M be a locally compact abelian group, and $G := M \times \widehat{M}$. The *Heisenberg multiplier* $\eta : (M \times \widehat{M}) \times (M \times \widehat{M})$ is given by

$$\eta\left((m, s), (n, t)\right) = \langle m, t \rangle.$$

Let D be a lattice in $M \times \widehat{M}$, and denote by $D^{\perp}$ the lattice in $M \times \widehat{M}$ given by

$$D^{\perp} = \left\{ (n, t) \in M \times \widehat{M} : \forall (m, s) \in D, \eta\left((m, s), (n, t)\right) \overline{\eta\left((n, t), (m, s)\right)} = 1 \right\}$$

Theorem (Rieffel '88)

Suitably completed under an appropriate norm, $C_c(M)$ is a $C^*(D, \eta)$-$C^*(D^{\perp}, \overline{\eta})$-equivalence bimodule.

Embed $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$ into $M \times \widehat{M}$

Let $M := \mathbb{Q}_p \times \mathbb{R}$, where $\mathbb{Q}_p$ denote the field of p-adic numbers.
$\mathbb{Q}_p$ is self-duel with the pairing $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{T}$, $\langle x, y \rangle = e^{2\pi i \{xy\}_p}$, where $\{x\}_p$ denotes the fractional part of x:

$$\left\{ \sum_{j=v}^{\infty} x_j p^j \right\}_p = \sum_{j=v}^{-1} x_j p^j .$$

Then M is self-dual.
We have $M = \widehat{M} = \mathbb{Q}_p \times \mathbb{R}$. For any $(x, \theta) \in [\mathbb{Q}_p \setminus \{0\}] \times [\mathbb{R} \setminus \{0\}]$, the map

$$\iota_{x,\theta} : \begin{cases} \mathbb{Z}[1/p] \times \mathbb{Z}[1/p] & \to \quad [\mathbb{Q}_p \times \mathbb{R}] \times [\mathbb{Q}_p \times \mathbb{R}] \\ (r_1 \times r_2) & \mapsto [(x \cdot r_1, \theta \cdot r_1), (r_2, r_2)] . \end{cases}$$

is an embedding, and we denote the image of $\iota_{x,\theta}$ by $D_{x,\theta}$. One can check that $D_{x,\theta}^{\perp} \cong \mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$.

Embed $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$ into $M \times \widehat{M}$

Theorem (Packer, Latrémolière, '13)

For every nonzero $\alpha \in \Xi_p$ with $x_0 \neq 0$, $C^*(D_{x,\theta}, \eta)$ is $*$-isomorphic to $\mathscr{A}_\alpha^{\mathscr{S}}$ for $x = x_\alpha$ and $\theta = \alpha_0$. Furthermore, $C^*(D_{x,\theta}^\perp, \overline{\eta})$ is $*$-isomorphic to a noncommutative solenoid.

($x = \sum_{j=0}^\infty x_j p^j$ is invertible in $\mathbb{Z}_p$ iff $x_0 \neq 0$.)

Proposition (L.)

Same set up as the theorem above, and let $x_\alpha^{-1} = \sum_{j=0}^\infty y_j p^j \in \mathbb{Z}_p$. Then $C^*(D_{x,\theta}^\perp, \overline{\eta})$ is $*$-isomorphic to $\mathscr{A}_\beta^{\mathscr{S}}$, where $\beta = (\beta_n)_{n\in\mathbb{N}}$ is given by

$$\beta_n = \frac{1}{\theta p^n} + \frac{\sum_{j=0}^{n-1} y_j p^j}{p^n}.$$

Remark: The Heisenberg bimodule construction gives the same Morita equivalence bimodules as building directed system of equivalence bimodules using the projection $P \in A_{\alpha_0}$ with $\tau_0(P) = \alpha_0$.

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Thank you!