Midterm

1. Let A be a point chosen uniformly at random from the circle $x^2 + y^2 = 1$. Compute the expectation of the distance from A to some fixed line through the origin.
2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1 \ldots dx_n \longrightarrow f(1/2)$$

   as $n \to \infty$. **Hint**: Interpret the left-hand side as an expected value; use the law of large numbers.
3. Let X and Y be iid random variables. Show that $X+Y \sim N(0, 2)$ if and only if both X and Y are standard normal random variables. (Prove this directly, do not use Cramér's decomposition theorem.)
4. Let $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ be sequences of random variables such that $X_n - Y_n \to 0$ in probability. Prove that if $X_n \to X$ in distribution, then $Y_n \to X$ in distribution.
5. Let $X_1, X_2, \ldots$ be independent random variables, and set

$$M_n = \max\{X_1, \ldots, X_n\}.$$

   Suppose X_n are identically distributed and their distribution function is $F(x) = 1 - x^{-\alpha}$ for $x > 1$ and $F(x) = 0$ for $x \leq 1$, with some $\alpha > 0$. Show that $M_n/n^{1/\alpha}$ converges in distribution, and find the limit.
6. A random variable X has a **lattice distribution** if there exists a real number a and a positive number b such that almost surely all values of X have the form $a + bn$, $n \in \mathbb{Z}$. Prove that X has a lattice distribution if and only if $|\phi_X(t)| = 1$ for some $t \neq 0$.
7. Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_k = k^{\alpha}) = \mathbb{P}(X_k = -k^{\alpha}) = \frac{1}{2},$$

   where $\alpha \geq -1/2$. State and prove a central limit theorem for these random variables.