Homework # 7

1. Let $X_1, \dots, X_n$ be iid standard normal random variables. Set $X = (X_1, \dots, X_n)^{\mathrm{T}}$.
   (a) Prove that UX has the same distribution as X for any deterministic $n \times n$ orthogonal matrix U.
   (b) Show that
   $$\frac{X}{\sqrt{X_1^2 + \cdots + X_n^2}}$$
   is uniformly distributed on the unit sphere in $\mathbb{R}^n$.
2. Let $X_1, \dots, X_n$ be iid standard normal random variables, and set $Z_n = \max_{1 \le i \le n} X_i$.
   (i) Show that the function $(x_1, \dots, x_n) \mapsto \max_{1 \le i \le n} x_i$ is 1-Lipschitz to deduce that, for any $\lambda > 0$,
   $$\mathbb{P}\left(|Z_n - \mathbb{E}Z_n| \ge \lambda\right) \le Ce^{-c\lambda^2},$$
   where $C, c > 0$ are absolute constants.
   (ii) Show that $\mathbb{E}Z_n \le \sqrt{2\log n}$ for all $n \ge 2$.
   (iii) Use the parts above to deduce that, for all $\lambda > 0$,
   $$\mathbb{P}\left(Z_n \ge \lambda + \sqrt{2\log n}\right) \le Ce^{-c\lambda^2},$$
   where $C, c > 0$ are absolute constants.
3. Let $X_1, \dots, X_n$ be iid standard normal random variables. Set
   $$S_n = \sum_{k=1}^{n} a_k X_k$$
   for some $a_1, \dots, a_n \in \mathbb{R}$. Show that if $\|a\|^2 = a_1^2 + \cdots + a_n^2 \ne 0$, then, for every $\varepsilon > 0$,
   $$\mathbb{P}\left(|S_n| \le \varepsilon\right) \le \frac{\varepsilon}{\|a\|}.$$
4. A graph G is **connected** when, for any two vertices x and y of G, there exists a sequence of vertices $x_0, x_1, \dots, x_k$ such that $x_0 = x$, $x_k = y$, and $x_i \sim x_{i+1}$ for $0 \le i \le k-1$. Show that the random walk on G is irreducible if and only if G is connected.