Homework # 6

1. Let τ be a stopping time with respect to the iid sequence $X_1, X_2, \ldots$. If $\mathbb{E}(\tau) < \infty$ and $\mathbb{E}|X_1| < \infty$, show that
$$\mathbb{E}\left(\sum_{n=1}^{\tau} X_n\right) = \mathbb{E}(\tau)\mathbb{E}(X_1).$$

2. If $(\mathcal{F}_n)_{n=1}^{\infty}$ is an increasing family of sub-σ-algebras, and $X \in L^1$ is a random variable, show that
$$\mathbb{E}(X|\mathcal{F}_n) \longrightarrow \mathbb{E}(X|\mathcal{F}_\infty)$$
almost surely as $n \to \infty$, where $\mathcal{F}_\infty = \sigma(\cup_{n=1}^{\infty} \mathcal{F}_n)$.

3. Let $X_1, \ldots, X_n$ be random variables such that $|X_k| \leq c_k$ almost surely for all $1 \leq k \leq n$. Assume
$$\mathbb{E}(X_k|X_1, \ldots, X_{k-1}) = 0$$
almost surely for $1 \leq k \leq n$. Show that, for any $\lambda > 0$, $S_n = \sum_{k=1}^n X_k$ satisfies
$$\mathbb{P}\left(|S_n| \geq \lambda \left(\sum_{k=1}^n c_k^2\right)^{1/2}\right) \leq C \exp(-c\lambda^2)$$
for some absolute constants $C, c > 0$.

4. Let X be a non-negative random variable with finite variance and $\mathbb{E}X \neq 0$. Show that for $a \in (0,1)$
$$\mathbb{P}(X \geq a\mathbb{E}X) \geq (1-a)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

5. Let X be random variable with mean zero. Define
$$\psi_X(t) = \log \mathbb{E}e^{tX}.$$
Assume X is non-degenerate (i.e. X is not almost surely a constant) and there exists $t_0 > 0$ such that $\psi_X(t_0) < \infty$.
   (i) Show that $\{t > 0 : \psi(t) < \infty\}$ is an interval, which we denote by $(0, b)$ or $(0, b]$.
   (ii) Find examples where the interval $\{t > 0 : \psi(t) < \infty\}$ is open and half-open.
   (iii) Show that ψ_X is convex and infinitely differentiable on $(0, b)$.