Homework # 5

1. Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ are sub-σ-algebras, prove that

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1)$$

and

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(X|\mathcal{G}_1).$$

2. Suppose S_n is a Gambler's fortune after n tosses of a fair coin, where the Gambler wins \$1 if comes up heads and loses \$1 if the coin comes up tails. Show that $T_n = S_n^2 - n$ is a martingale.

3. Let X and Y be iid random variables with finite mean. Show that

$$\mathbb{E}(X|X+Y) = \frac{X+Y}{2}.$$

4. Define the **conditional variance** as

$$\mathrm{Var}(X|\mathcal{F}) = \mathbb{E}(X^2|\mathcal{F}) - \mathbb{E}(X|\mathcal{F})^2.$$

Show that

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\mathcal{F})) + \mathrm{Var}(\mathbb{E}(X|\mathcal{F})).$$

5. Prove that if $X_n \to X$ in L^p as $n \to \infty$ for some $1 \le p \le \infty$, then there exists a subsequence $(X_{n_k})_{k=1}^{\infty}$ such that $X_{n_k} \to X$ almost surely as $k \to \infty$.

6. (**Bonus Problem**) This problem was motivated by a question Katy Smith asked. Let X be an integrable random variable. Is it true that

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) = \mathbb{E}(X|\mathcal{G}_1 \cap \mathcal{G}_2)$$

for all sub-σ-algebras $\mathcal{G}_1, \mathcal{G}_2$? If yes, prove the statement; if not, give a counter-example.