Homework # 4

1. Let $X_1, X_2, \ldots$ be iid uniform random variables on $(0,1)$. For all $n \geq 1$, define the random variable $M_n = \min\{X_1, \ldots, X_n\}$. Show that

$$nM_n \longrightarrow \mathrm{Exp}(1)$$

   in distribution as $n \to \infty$.

2. Show that the sequence of random variables $(X_n)_{n=1}^{\infty}$ is tight if and only if for all $\varepsilon > 0$ there exists $M > 0$ such that

$$\mathbb{P}(|X_n| \geq M) < \varepsilon$$

   for all $n \geq 1$.

3. Show that if $(X_n)_{n=1}^{\infty}$ is a tight sequence of random variables such that every convergent subsequence $(X_{n_k})_{k=1}^{\infty}$ has the same distributional limit X, then the sequence $(X_n)_{n=1}^{\infty}$ converges in distribution to X.

4. Compute the characteristic functions for the following distributions.

   (a) Poisson distribution: For $\lambda > 0$, we say X has Poisson distribution with parameter λ (i.e. $X \sim \mathrm{Poisson}(\lambda)$) if, for $k = 0, 1, 2, \ldots$,

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

   (b) Geometric distribution: $X \sim \mathrm{Geom}(p)$.

   (c) Uniform distribution: $X \sim \mathrm{Uniform}[-1, 1]$.

   (d) Exponential distribution: $X \sim \mathrm{Exp}(\lambda)$.

5. Prove the following.

   (a) If X is a random variable, then $\mathrm{Re}(\varphi_X)$ and $|\varphi_X|^2$ are also characteristic functions (i.e. construct random variables Y and Z such that $\varphi_Y(t) = \mathrm{Re}(\varphi_X(t))$, $\varphi_Z(t) = |\varphi_X(t)|^2$).

   (b) We say X is **symmetric** if X is equal in distribution to $-X$. Show that X is symmetric if and only if φ_X is a real-valued function.