Homework # 3

1. Let X be a random variable. Prove that, for any $\varepsilon > 0$, there exists a bounded random variable X_ε such that
$$\mathbb{P}(X \neq X_\varepsilon) < \varepsilon.$$

2. Show that, for any random variable X and any $a > 0$, one has
$$\int_{\mathbb{R}} \mathbb{P}(x < X < x + a) dx = a.$$
Hint: Apply Fubini's theorem.

3. Let $X_1, X_2, \ldots$ be iid random variables with distribution
$$\mathbb{P}(X_1 = 2^k) = 2^{-k}, \quad k = 1, 2, 3, \ldots.$$
Set $S_n = \sum_{k=1}^n X_k$. Show that:
   (a) $\mathbb{E}X_1 = \infty$, and hence the WLLN does not apply to $\frac{1}{n} S_n$,
   (b) one has
$$\frac{S_n}{n \log_2 n} \xrightarrow{P} 1$$
   as $n \to \infty$.

   Hint: For part (3b), it is helpful to truncate the first n random variables at a level that depends on n. That is, denote $b_n = n \log_2 n$, $Y_{n,k} = X_k \mathbf{1}_{\{X_k < b_n\}}$, $T_n = \sum_{k=1}^n T_{n,k}$, and $a_n = \mathbb{E}T_n$. You will want to show that
$$\frac{T_n - a_n}{b_n} \xrightarrow{P} 0$$
   as $n \to \infty$.

4. Let $(c_n)_{n=1}^\infty$ be a sequence of real numbers. Consider the sum $\sum_{n=1}^\infty c_n X_n$, where $X_1, X_2, \ldots$ is a sequence of iid symmetric Bernoulli random variables (i.e. X_1 takes values ± 1 with probability $1/2$). Show that if $\sum_{n=1}^\infty c_n^2 < \infty$, then the series $\sum_{n=1}^\infty c_n X_n$ converges almost surely. As a bonus, try to prove the converse.

5. Let $X_0 = (1, 0)$, and define $X_n \in \mathbb{R}^2$ inductively by declaring that X_{n+1} is chosen at random (uniformly) from the ball of radius $|X_n|$ centered at the origin (i.e. $X_{n+1}/|X_n|$ is uniformly distributed on the ball of radius one and independent of $X_1, \ldots, X_n$). Prove that
$$\frac{1}{n} \log |X_n| \xrightarrow{a.s.} c$$
as $n \to \infty$ and compute c.