Homework # 2

1. We say one random variable Y is *determined* by another random variable X if there exists a (Borel) measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$ (i.e. $Y(\omega) = f(X(\omega))$ for all ω in the sample space Ω). Let X, Y be random variables. Show that Y is a constant (i.e. a constant function with probability one) if and only if it is simultaneously determined by X, and independent of X.
2. For each $n \geq 1$, let X_n be a random variable such that
$$\mathbb{P}(X_n = n) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2}.$$
Show that $\sum_{n=1}^{\infty} X_n < \infty$ almost surely, but $\sum_{n=1}^{\infty} \mathbb{E}(X_n) = \infty$.
3. Prove Jensen's inequality: if $\phi : \mathbb{R} \to \mathbb{R}$ is convex then
$$\phi(\mathbb{E}X) \leq \mathbb{E}(\phi(X)).$$
Hint: Recall that an affine linear function is a function of the form $x \mapsto ax + b$ for some constants $a, b \in \mathbb{R}$. You may use the following fact without proof. If $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then, for every $x_0 \in \mathbb{R}$, there exists an affine linear function l_{x_0} such that
$$\phi(x) \geq l_{x_0}(x)$$
for every $x \in \mathbb{R}$, and $\phi(x_0) = l_{x_0}(x_0)$.
4. Give an example to show that convergence in probability does not imply almost sure convergence. **Hint**: Consider the following sequence of random variables defined on the probability space $((0, 1), \mathcal{B}, \lambda)$:
$$\begin{aligned}
&\mathbf{1}_{(0,1)},\\
&\mathbf{1}_{(0,1/2)}, \mathbf{1}_{(1/2,1)}\\
&\mathbf{1}_{(0,1/4)}, \mathbf{1}_{(1/4,1/2)}, \mathbf{1}_{(1/2,3/4)}, \mathbf{1}_{(3/4,1)}\\
&\vdots
\end{aligned}$$
5. Let $X, X_1, X_2, \ldots$ be random variables. Show that if X_n converges in distribution to X and F_X is continuous, then
$$\sup_{x \in \mathbb{R}} |F_{X_n}(x) - F_X(x)| \to 0.$$