Homework # 1

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that
   (i) If $A, B \in \mathcal{F}$, then
   $$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$
   (ii) If $A_1, A_2, \ldots \in \mathcal{F}$ such that $A_1 \subset A_2 \subset A_3 \subset \cdots$, then
   $$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mathbb{P}(A_n).$$
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A \in \mathcal{F}$. Show that the set of all $B \in \mathcal{F}$ which satisfy
   $$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$
   is a λ-system.
3. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces such that $\mathcal{F}_2$ is the σ-algebra generated by a collection $\mathcal{A}$ of subsets of Ω_2. Prove that a function $X : \Omega_1 \to \Omega_2$ is measurable if and only if $X^{-1}(A) \in \mathcal{F}_1$ for all $A \in \mathcal{A}$.
4. Let X be a random variable. Show that μ_X is a probability measure on $(\mathbb{R}, \mathcal{B})$.
5. Let X be a random variable, and suppose $F(x) = \mathbb{P}(X \leq x)$ is continuous. Show that $Y = F(X)$ has a uniform distribution on $(0, 1)$.