Final Exam

1. Let X and Y be random variables uniformly distributed on $(0,1)$. Show that, whatever the dependence between X and Y, one has
$$\mathbb{E}|X-Y| \leq \frac{1}{2}.$$

2. Show that
$$L(F,G) = \inf\{\varepsilon > 0 : F(x-\varepsilon) - \varepsilon \leq G(x) \leq F(x+\varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\}$$
defines a metric on the space of cumulative distribution functions and $L(F_n, F) \to 0$ is equivalent to convergence in distribution. This is the **Lévy metric**.

3. Suppose X and Y are random variables which satisfy
$$\mathbb{E}(\max\{X,Y\} | \min\{X,Y\}) = \min\{X,Y\}.$$
Show that $X = Y$ almost surely.

4. Let P be the transition matrix of an irreducible Markov chain with state space Ω. Let $B \subset \Omega$ be a non-empty subset of the state space, and assume $h : \Omega \to \mathbb{R}$ is a function harmonic at all states $x \notin B$. Prove that there exists $y \in B$ with $h(y) = \max_{x\in\Omega} h(x)$.

5. Show that a random walk on a finite group is reversible if and only if the increment distribution is symmetric.

6. Consider a random walk on the set $\{0, 1, \ldots, n\}$ in which the walk moves left or right with equal probability except when at n and 0. At n, it remains at n with probability 1/2 and moves to $n-1$ with probability 1/2, and once the walk hits 0, it remains there forever. Compute the expected time of the walk's absorption at state 0, given that it starts at state n.

7. A countably infinite number of prisoners are forced by an evil warden to play the following game. Each prisoner will be randomly assigned a hat that is either black or white; he can see all the other prisoners' hats but not his own. Each prisoner must try to guess the color of his own hat. The prisoners may strategize before the game, but once the hats are assigned, they may not communicate in any way, and they do not get to hear each others' guesses. If all but finitely many of the prisoners can guess correctly, all the prisoners will be freed; otherwise, they will all be executed.
Formally, let X_n be the hat assigned to prisoner n, 0 for black and 1 for white. Assume the warden chooses hats randomly by flipping a fair coin. That is, $(X_n)_{n=1}^{\infty}$ is a sequence of iid random variables with $\mathbb{P}(X_n = 0) = 1/2 = \mathbb{P}(X_n = 1)$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{G}_n = \sigma(X_k : k \neq n)$. Let Y_n be the n-th prisoner's guess. He can see every hat except his own, so Y_n is $\mathcal{G}_n$-measurable. (Note that the guesses $(Y_n)_{n=1}^{\infty}$ need not be independent; for instance, prisoners 2 and 3 could both base their guesses on the color of prisoner 1's hat.)
Let $A_n = \{Y_n = X_n\}$ be the event that prisoner n guesses correctly, and $A = \liminf A_n$ is the event that all but finitely many prisoners guess correctly. Show, unfortunately, that $\mathbb{P}(A) = 0$ using the following steps.
   (i) Show A_n is independent of $\mathcal{G}_n$ (even though Y_n is not), and that $\mathbb{P}(A_n) = 1/2$.
   (ii) Prove that, for $n > m$, $\mathbb{E}[\mathbf{1}_{A_n} | \mathcal{F}_m] = 1/2$ almost surely.
   (iii) Show that $\mathbb{E}[\mathbf{1}_A | \mathcal{F}_m] \leq 1/2$ almost surely.
   (iv) Conclude that $\mathbb{P}(A) = 0$.
   (v) For a sense of why this may not be completely obvious, consider the case where there are only a finite number of prisoners. Give an example of a strategy such that, with probability 1/2, all of them guess correctly. (That is, using the notation from above, if there are N prisoners, find random variables $(Y_n)_{n=1}^{N}$ such that each Y_n is $\mathcal{G}_n$-measurable and $\mathbb{P}(\cap_{n=1}^{N} A_n) = 1/2$.)