Homework # 4

1. Let X and Y be metric spaces. If $\{f_n\}$ is a sequence of continuous functions from X to Y which converges uniformly to $f : X \to Y$, prove that f must be continuous.
2. Let X be a compact metric space. Assume $\{f_n\}$ is a monotonically increasing sequence (meaning $f_1 \leq f_2 \leq f_3 \leq \cdots$) of continuous real-valued functions on X which converges pointwise to a continuous function $f : X \to \mathbb{R}$. Prove that $\{f_n\}$ converges to f uniformly.
3. Suppose $\{f_n\}$ is a sequence of functions in $L^+(X, \mathcal{A})$ such that $f_n \to f$ pointwise and $\int f = \lim_{n\to\infty} \int f_n < \infty$. Show that $\int_A f = \lim_{n\to\infty} \int_A f_n$ for all $A \in \mathcal{A}$.
4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and take $f \in L^+(X, \mathcal{A})$. If $\int f \; d\mu < \infty$, prove that $\{x \in X : f(x) = \infty\}$ is a μ-null set.
5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and take $f \in L^+(X, \mathcal{A})$. Show that $\nu(A) = \int_A f \; d\mu$ is a measure on $\mathcal{A}$ and that
$$\int g \; d\nu = \int fg \; d\mu$$
for every $g \in L^+(X, \mathcal{A})$.