Homework # 3

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $g \circ f$ is Lebesgue measurable.
2. Let $(X, \mathcal{A})$ be a measurable space. Let $\{f_n\}$ be a sequence of measurable real-valued functions on X. Show that
$$A = \left\{ x \in X : \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$$
is an $\mathcal{A}$-measurable set.
3. Let X be the integers with the discrete metric
$$\rho(m, n) = \begin{cases} 1, & \text{if } m \neq n, \\ 0, & \text{if } m = n. \end{cases}$$
Check that ρ is a metric. Show that X is closed and bounded, but not compact.
4. Prove that every compact subset of a metric space is closed and bounded. Prove that a closed subset of a compact space is compact.
5. A metric space (X, ρ) is said to be an *ultrametric space* if
$$\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$$
for all $x, y, z \in X$. Prove that, in an ultrametric space, every open ball
$$B(x, r) = \{y \in X : \rho(x, y) < r\}$$
is also closed.
6. A metric space (X, ρ) is said to be *perfect* if every point in X is an accumulation point, meaning that it is the limit of a sequence of other points in the space.
Let $X = \{0, 1\}^{\mathbb{N}}$ be the space of all sequences consisting of zeros or ones:
$$X = \{(s_1, s_2, s_3, \ldots) : s_n \in \{0, 1\}\}.$$
Define $\rho : X \times X \to [0, \infty)$ by
$$\rho(\mathbf{s}, \mathbf{t}) = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n},$$
where $\mathbf{s} = (s_1, s_2, s_3, \ldots)$, $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, and
$$\delta_n = \begin{cases} 0, & \text{if } s_n = t_n, \\ 1, & \text{if } s_n \neq t_n. \end{cases}$$
   (a) Prove that ρ is a metric on X.
   (b) Show that X is compact and perfect.
   (c) Define the shift map $\sigma : X \to X$ by
   $$\sigma(s_1, s_2, s_3, \ldots) = (s_2, s_3, s_4, \ldots).$$
   Prove that σ is continuous. Is σ uniformly continuous?