Solutions to Homework #11

1. (a) Let f be differentiable on $[a, b]$ and suppose f' is continuous on $[a, b]$. By the Extreme Value Theorem, there exist $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. We will prove that

$$|f(y) - f(x)| \leq M|y - x|$$

for all $x, y \in [a, b]$. Let $x, y \in [a, b]$. If $x = y$, then clearly $|f(y) - f(x)| = 0 = M|y - x|$. Suppose $x \neq y$, and without loss of generality, assume $x < y$. Then, by the Mean Value Theorem applied to the interval $[x, y]$, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x).$$

Thus, we obtain

$$|f(y) - f(x)| = |f'(c)||y - x| \leq M|y - x|.$$

We conclude that f is Lipschitz on $[a, b]$.

(b) The conclusion follows from part (a) by noting that $f'(x) = \cos(x)$ is continuous.

3. (a) $f(x) = |x|$ is Lipschitz on $\mathbb{R}$, but not differentiable at zero.

(b) Take

$$f(x) = \begin{cases} 0, & \text{if } x < 1/2, \\ 1, & \text{if } x \geq 1/2. \end{cases}$$

(c) The function f from part (b) is increasing, but not differentiable (since its not continuous at $x = 1/2$).

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. We will prove the contrapositive statement. To that end, assume f is not one-to-one. Then there exits $x, y \in \mathbb{R}$ with $x < y$ such that $f(x) = f(y)$. By the Mean Value Theorem applied to the interval $[x, y]$, there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = 0.$$