Practice Final

The following is a list of problems I consider final-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the final.

1. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, \ldots$. Prove that K is compact directly from the definition.
2. Use the ε–δ definition of functional limits to how that
$$\lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4}.$$
3. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that f has a fixed point. That is, prove that there exists $c \in [a, b]$ such that $f(c) = c$.
4. Let $a_n = \sqrt{n^2 + n} - n$. Calculate $\lim(a_n)$.
5. Define $a_1 = 1$ and
$$a_{n+1} = \sqrt{1 + \sqrt{a_n}}$$
for $n \in \mathbb{N}$. Does (a_n) converge? If so, what is the limit?
6. Assume $f : [a, b] \to \mathbb{R}$ is integrable.
   (a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a, b]$, then g is integrable as well.
   (b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.
7. Prove the mean value theorem for integrals: If f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that
$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx.$$
8. Prove the inequality:
$$\frac{x}{1 + x} \leq \ln(1 + x) \leq x \text{ for all } x > -1.$$
9. Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that f is not integrable on $[0, 1]$, but f^2 is integrable on $[0, 1]$.
10. For $x, y \in \mathbb{R}$, define
$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$
Determine, for each of these, whether it is a metric or not.
11. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.
12. Suppose f is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
13. Find an open cover of $\mathbb{R}$ that has no finite subcover.

(**Bonus**) A real-valued function f defined in (a, b) is said to be **convex** if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Prove that every convex function is continuous.