Select solutions to Homework #8

19.4 For each part below, I will simply state the set S of sub-sequential limits. The limit superior and limit inferior easily follow from S and Definition 19.9. Formally, Definition 19.9 implies that S is always a set of real numbers. However, following the practice problems (see Practice 19.15), we will include $+\infty$ (alternatively, $-\infty$) if there exists a subsequence which diverges to $+\infty$ (alternatively, $-\infty$).

(a) $S = \{0\}$ since (s_n) converges to 0.

(b) $S = \{0, 1, +\infty\}$.

(c) $S = \{+\infty\}$.

(d) $S = \{+\infty, -\infty\}$.

19.9 Assume $\limsup s_n = s = \liminf s_n$. Then, by Definition 19.9, it follows that $S = \{s\}$, where S is the set of sub-sequential limits of (s_n). Assume (s_n) does not converge to s. Then there exists $\varepsilon > 0$ and a subsequence $(s_{n_k})_{k=1}^{\infty}$ of (s_n) such that

$$|s_{n_k} - s| > \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

By Theorem 19.7, there exists a further subsequence $(s_{n_{k_l}})_{l=1}^{\infty}$ that converges. Since $(s_{n_{k_l}})_{l=1}^{\infty}$ is a subsequence of (s_n), it must be the case that $(s_{n_{k_l}})_{l=1}^{\infty}$ converges to s (recall that $S = \{s\}$). This contradictions (1). Therefore, we conclude that $\lim s_n = s$.

20.4 Take $\delta = 1/36$. Then,

$$|x^2 + 2x - 15| \leq |x - 3||x + 5| < \delta|x + 5| = \frac{1}{36}|x + 5|.$$

Thus, it suffices to show $|x + 5| \leq 9$. Indeed, since $\delta < 1$,

$$|x + 5| = |x - 3 + 8| \leq |x - 3| + 8 \leq 1 + 8 = 9,$$

and the proof is complete.