Select solutions to Homework #7

17.8 (a) Consider $s_n = 1$.

(b) Take $t_n = n$.

17.9 Suppose $\lim s_n = +\infty$ and $s_n \leq t_n$ for all $n \in \mathbb{N}$. Then, for every M, there exists N such that $s_n > M$ whenever $n > N$. Thus, for $n > N$, we obtain $t_n \geq s_n > M$. Hence, we conclude that $\lim t_n = +\infty$. The proof of part (b) is similar.

17.16 Let c be a real number such that $1 < c < L$ (such a choice is always possible by Theorem 12.12). Set $\varepsilon = L - c > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\left| \frac{s_{n+1}}{s_n} - L \right| < \varepsilon$$

whenever $n > N$. Thus, for $n > N$

$$\frac{s_{n+1}}{s_n} = \left| \frac{s_{n+1}}{s_n} \right| \geq L - \left| \frac{s_{n+1}}{s_n} - L \right| \geq L - \varepsilon = c$$

by the reverse triangle inequality. Therefore, for $n > N$, we conclude that $s_{n+1} \geq c s_n$. Define $k = N + 1$. Then, for $n > k$, we iterate the previous inequality to obtain

$$s_n \geq c s_{n-1} \geq \cdots \geq c^{n-k} s_k.$$

Define $M = c^{-k} s_k > 0$. Then we have shown that, for $n > k$,

$$s_n \geq M c^n.$$

Since $\lim c^n = +\infty$, it follows that $\lim s_n = +\infty$ by Theorem 17.12.

18.4 (a) $s_n = (-1)^n \frac{1}{n}$.

(b) $t_n = n$.

(c) $u_n = (-1)^n$.