Select solutions to Homework #6

15.10 Suppose A and B are compact. Let $\mathcal{F}$ be an open cover of $A \cup B$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows that $\mathcal{F}$ is an open cover of A and an open cover of B. Since A is compact, there exists a finite sub-cover $\mathcal{G}_1 \subseteq \mathcal{F}$ of A. Similarly, since B is compact, there exists a finite sub-cover $\mathcal{G}_2 \subseteq \mathcal{F}$ of B. Then $\mathcal{G}_1 \cup \mathcal{G}_2$ is a sub cover of $A \cup B$. Moreover, since $\mathcal{G}_1$ and $\mathcal{G}_2$ are finite, it follows that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a finite sub-cover of $A \cup B$. Hence $A \cup B$ is compact.

15.16 (a) Suppose D is dense. In order to reach a contradiction, assume $A \subseteq X$ is a nonempty open set that does not intersect D. Since A is open, there exists $x \in A$ and $\varepsilon > 0$ such that $N(x, \varepsilon) \subseteq A$. Thus, $N(x, \varepsilon)$ does not intersect D, and hence x is not in D and x is not an accumulation point of D. Therefore, $x \notin \overline{D}$. This contradicts the assumption that D was dense.

Conversely, assume every nonempty open subset of X intersects D. Let $x \in X$. If $x \in D$, then $x \in \overline{D}$. Suppose $x \notin D$. By Theorem 15.6 and our supposition, it follows that $N(x, \varepsilon)$ intersects D for every $\varepsilon > 0$. Since $x \notin D$, it must be the case that $N^*(x, \varepsilon)$ intersects D for every $\varepsilon > 0$. Hence, x is an accumulation point of D. Thus, we conclude that D is dense.

16.9 (a) As $||s_n| - |s|| \leq |s_n - s|$ by the reverse triangle inequality, part (a) follows from Theorem 16.8.

(b) Consider the following counterexample: $s_n = (-1)^n$.

(c) Follows from Definition 16.2.

16.13 By assumption, $a_n - b \leq b_n - b \leq c_n - b$ for all $n \in \mathbb{N}$. Thus,

$$|b_n - b| \leq \max\{|a_n - b|, |c_n - b|\} \tag{1}$$

by definition of absolute value. Let $\varepsilon > 0$. Since $\lim a_n = b$ and $\lim c_n = b$, there exists N_1 and N_2 such that $|a_n - b| < \varepsilon$ whenever $n > N_1$ and $|c_n - b| < \varepsilon$ whenever $n > N_2$. Take $N = \max\{N_1, N_2\}$. Then, for $n > N$, we have

$$|b_n - b| \leq \max\{|a_n - b|, |c_n - b|\} < \varepsilon$$

by (1). Therefore, we conclude that $\lim b_n = b$.

17.7 Consider $s_n = (-1)^n n$.