Select solutions to Homework #3

8.8 (a) Assume $S \subseteq T$. Define $f : S \to T$ by $f(x) = x$. Since f is injective by construction, $|S| \leq |T|$.

(b) Follows from part (a).

(c) Assume $|S| \leq |T|$ and $|T| \leq |U|$. Then there exists injections $f : S \to T$ and $g : T \to U$. By Theorem 7.19, $g \circ f : S \to U$ is injective. Thus, $|S| \leq |U|$.

(d) Follows form part (a).

(e) Suppose S is a finite nonempty set. Then there exists $n \in \mathbb{N}$ and a bijection $f : I_n \to S$. It follows that $f^{-1} : S \to \mathbb{N}$ is injective, and so $|S| \leq |\mathbb{N}|$. We now show $|S| \neq |\mathbb{N}|$. Suppose, to the contrary, that $|S| = |\mathbb{N}|$. In other words, suppose there exists a bijection $g : S \to \mathbb{N}$. Then, by Theorem 7.19, $g \circ f : I_n \to \mathbb{N}$ is a bijection. This implies that $\mathbb{N}$ is finite, a contradiction. We conclude that $|S| < |\mathbb{N}|$. The case when S is the empty set is trivial.

8.17 If $A \subseteq B$, then any subset S of A is also a subset of B since $S \subseteq A \subseteq B$. Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Conversely, if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then

$$A \in \mathcal{P}(A) \subseteq \mathcal{P}(B),$$

and hence $A \subseteq B$.

8.19 (a) Assume $|S| \leq |T|$. Then there exists an injection $f : S \to T$. Define $g : \mathcal{P}(S) \to \mathcal{P}(T)$ by

$$g(A) = f(A) \quad \text{for all } A \in \mathcal{P}(S).$$

It suffices to show that g is injective. That is, we will show that if $g(A) = g(B)$ for some sets $A, B \in \mathcal{P}(S)$, then $A = B$. Suppose $g(A) = g(B)$. By definition of g, this implies that $f(A) = f(B)$. From Theorem 7.17, we conclude that $A = B$, and hence g is injective.

10.26 Let

$$N = \{n \in \mathbb{N} : \text{ there exists } k \text{ such that } n \leq k^2 \leq 2n\}.$$

Clearly $1 \in N$. Suppose $n \in N$. Then there exists an integer k such that $n \leq k^2 \leq 2n$. If $k^2 > n$, then $n + 1 \leq k^2 \leq 2(n + 1)$. Thus, it suffices to consider the case when $k^2 = n$. In this case,

$$n + 1 = k^2 + 1 \leq (k + 1)^2 \leq 2(k^2 + 1) = 2(n + 1)$$

by Exercise 4.25. Therefore, $n + 1 \in N$. By the principle of induction, we conclude that $N = \mathbb{N}$.