Select solutions to Homework #2

7.16 See Example 7.13 in the text for several examples.

7.18 (a) By Theorem 7.15,
$$C \subseteq f^{-1}[f(C)].$$
Thus, it remains to show the reverse inclusion. Let $x \in f^{-1}[f(C)]$. Then, by definition of the pre-image, $f(x) \in f(C)$. This implies, by definition of the image, that $f(x) = f(c)$ for some $c \in C$. As f is injective, we conclude that $x = c \in C$, and hence $x \in C$. Therefore, it follows that
$$C = f^{-1}[f(C)].$$

(b) By Theorem 7.15,
$$f(f^{-1}(D)) \subseteq D. \tag{1}$$
Thus, it remains to show the reverse inclusion. Let $y \in D \subseteq B$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. In other words, $x \in f^{-1}(D)$. Since $f(x) = y$, this implies that $y \in f(f^{-1}(D))$, and thus
$$D \subseteq f(f^{-1}(D)).$$
In view of (1), we conclude that
$$D = f(f^{-1}(D)).$$

7.33 (b) Let C be an arbitrary element of E. Then $C = E_x$ for some $x \in A$. Thus, by definition of the function g, we have
$$g(x) = E_x = C.$$
Since $C \in E$ was arbitrary, we conclude that g is surjective. [Note: this function need not be injective.]

10.4 Define
$$N = \left\{ n \in \mathbb{N} : 1^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 \right\}.$$
It easily follows that $1 \in N$. Suppose $n \in N$. Then
$$\begin{aligned} 1^3 + \cdots n^3 + (n+1)^3 &= \frac{1}{4}n^2(n+1)^2 + (n+1)^3 \\ &= \frac{1}{4}(n+1)^2\left(n^2 + 4n + 4\right) \\ &= \frac{1}{4}(n+1)^2(n+2)^2. \end{aligned}$$
Thus, we conclude that $n + 1 \in N$. By the principle of induction, $N = \mathbb{N}$, and the proof is complete. [Observe that the solution to this problem immediately verifies Exercise 10.5 as well.]