Select solutions to Homework #10

21.10 (a) Let $\varepsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in D$. Thus, by the reverse triangle inequality, for $|x - c| < \delta$ and $x \in D$, we obtain

$$||f(x)| - |f(c)|| \leq |f(x) - f(c)| < \varepsilon.$$

Hence, $|f|$ is continuous at c.

(b) No. Consider

$$f(x) = \begin{cases} -1, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

22.8 Define $h : [a, b] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. By Theorem 21.10, it follows that h is continuous. By supposition, we find that $h(a) = f(a) - g(a) \leq 0$ and $h(b) = f(b) - g(b) \geq 0$. If $h(a) = 0$, then we can take $c = a$. Similarly, if $h(b) = 0$, take $c = 0$. Thus, it suffices to consider the case where $h(a) \neq 0$ and $h(b) \neq 0$. In this setting, the intermediate value theorem implies that there exists $c \in (a, b)$ such that $h(c) = f(c) - g(c) = 0$. In any case, we conclude that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

22.9 Assume f is non-constant on $[a, b]$. Then there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \neq f(x_2)$. In particular, this implies that $x_1 \neq x_2$. By Theorem 12.14, there exists an irrational number k between $f(x_1)$ and $f(x_2)$. Thus, by the intermediate value theorem, there exists c between x_1 and x_2 such that $f(c) = k$; this contradicts the assumption that $f([a, b]) \subseteq \mathbb{Q}$.

23.4 (a) Let $\varepsilon > 0$. Take $\delta = \varepsilon/12$. Then, for $x, y \in [0, 2]$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |x^3 - y^3| = |x - y||x^2 + xy + y^2| < \delta|x^2 + xy + y^2| \leq 12\delta = \varepsilon.$$

Here we used that $|x^2 + xy + y^2| \leq 12$, by the triangle inequality, whenever $x, y \in [0, 2]$.

(b) Let $\varepsilon > 0$. Take $\delta = \varepsilon/4$. Then, for $x, y \in [2, \infty)$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| = \left| \frac{x - y}{xy} \right| \leq \frac{1}{4}|x - y| < \frac{1}{4}\delta = \varepsilon.$$