Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. Consider the set $\mathbb{R}^2$. Define the metrics d_1 and d_2 on $\mathbb{R}^2$ by

$$d_1((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|,$$
$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

   (a) Draw a picture depicting the neighborhood $N((0,0),1)$ in the metric space $(\mathbb{R}^2, d_1)$.

   (b) Draw a picture depicting the neighborhood $N((0,0),1)$ in the metric space $(\mathbb{R}^2, d_2)$.

   (c) Explain the differences between the two pictures.

2. Let (s_n) be the sequence defined by $s_1 = \sqrt{3}$ and $s_{n+1} = \sqrt{3 + s_n}$ for all $n \in \mathbb{N}$. Does (s_n) converge? If so, what is the limit of (s_n)?

3. Use the ε–δ definition of convergence to prove the following:

$$\lim_{x \to 4} f(x) = f(4),$$

   where $f(x) = x^2 - x + 5$.

4. Let (s_n) be a sequence of real numbers. Define the sequence (σ_n) by

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}$$

   for all $n \in \mathbb{N}$.

   (a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

   (b) Give an example of a sequence (s_n) which does not converge, but for which (σ_n) converges to zero.

5. Prove or give a counter example: If $f : D \to \mathbb{R}$ is not continuous and $g : D \to \mathbb{R}$ is not continuous, then fg is not continuous on D.

6. Let $s \in \mathbb{R}$. Let (s_n) be a sequence of real numbers which satisfies the following property: every subsequence of (s_n) has a further subsequence that converges to s. Prove that $\lim s_n = s$.

7. Use the ε–δ definition of convergence to prove the following: if $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} f(x)^2 = L^2$. Is the converse true?

8. Let (a_n) be a bounded sequence. Suppose there exists a sequence of integers $(n_k)_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} n_k = +\infty$, $\lim_{k\to\infty} n_k/n_{k+1} = 1$, and

$$\lim_{k\to\infty} \frac{a_1 + \cdots + a_{n_k}}{n_k} = a.$$

   Prove that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = a.$$

9. (**Bonus**) Suppose (p_n) and (q_n) are Cauchy sequences in a metric space (X, d). Show that the sequence $(d(q_n, p_n))$ converges.