Practice Midterm

The following is a list of problems I consider midterm-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the midterm.

1. Let R and S be relations on a set A. Prove or give a counterexample for each of the following.
   (a) If R and S are reflexive, then $R \cap S$ is reflexive.
   (b) If R and S are reflexive, then $R \cup S$ is reflexive.
   (c) If R and S are symmetric, then $R \cap S$ is symmetric.
   (d) If R and S are symmetric, then $R \cup S$ is symmetric.
   (e) If R and S are transitive, then $R \cap S$ is transitive.
   (f) If R and S are transitive, then $R \cup S$ is transitive.
2. Use induction to prove that $2 + 5 + 8 + \cdots + (3n - 1) = \frac{1}{2}n(3n + 1)$ for all $n \in \mathbb{N}$.
3. Let $x \in \mathbb{R}$. Prove that $x = \sup\{q \in \mathbb{Q} : q < x\}$.
4. Let S be a subset of $\mathbb{R}$.
   (a) Prove that $\operatorname{int} S$ is always open.
   (b) If $T \subset S$ and T is open, prove that $T \subset \operatorname{int} S$.
   (c) Do S and $\bar{S}$ always have the same interiors?
5. Define a relation R on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $(a, b)R(x, y)$ iff $ay = bx$.
   (a) Prove that R is an equivalence relation.
   (b) Describe the equivalence classes corresponding to R.
6. Use induction to prove that if $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.
7. Give an example of an open cover of $(0, 1)$ which has no finite sub-cover.
8. Prove that every infinite set is equinumerous with a proper subset of itself.
9. Let S be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is S countable? Is S compact?
10. Prove that the intersection of any collection of compact sets is compact.
11. Suppose that $g : A \to C$ and $h : B \to C$. Prove that if h is bijective, then there exists a function $f : A \to B$ such that $g = h \circ f$.
12. Use the definition of compactness to show that $\mathbb{R}$ is not compact.