Practice Final

The following is a list of problems I consider final-worthy. This list of problems should serve as a good place to start studying, and it should not be considered a comprehensive list of problems from the sections we've covered. YOU are responsible for studying all the sections to be covered on the final.

1. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, \ldots$. Prove that K is compact directly from the definition.
2. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that f has a fixed point. That is, prove that there exists $c \in [a, b]$ such that $f(c) = c$.
3. Let $s_n = \sqrt{n^2 + n} - n$. Calculate $\lim s_n$.
4. Let $\alpha > 0$. Define $s_1 = \sqrt{\alpha}$ and
$$s_{n+1} = \sqrt{\alpha + \sqrt{s_n}}$$
for $n \in \mathbb{N}$. Does (s_n) converge? If so, what is the limit?
5. Let $C > 0$ and let I be an interval. A function $f : I \to \mathbb{R}$ is called **C-Lipschitz continuous** on I if $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in I$. A function $f : I \to \mathbb{R}$ is called **Lipschitz continuous** on I if there exists $C > 0$ such f is C-Lipschitz continuous on I.
   (a) Show that every Lipschitz continuous function on I is uniformly continuous on I.
   (b) Show that $\sin(x)$ is Lipschitz continuous on $\mathbb{R}$.
   (c) Give an example of a Lipschitz continuous function on $\mathbb{R}$ which is not differentiable on $\mathbb{R}$.
6. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by
$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Prove the following:
   (a) f is continuous if and only if $a > 0$.
   (b) $f'(0)$ exists if and only if $a > 1$.
   (c) f' is bounded if and only if $a \geq 1 + c$.
   (d) f' is continuous if and only if $a > 1 + c$.
   (e) $f''(0)$ exists if and only if $a > 2 + c$.
7. Prove the mean value theorem for integrals: If f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that
$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx.$$
8. Prove the inequality:
$$\frac{x}{1 + x} \leq \ln(1 + x) \leq x \text{ for all } x > -1.$$
9. For $x, y \in \mathbb{R}$, define
$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$
Determine, for each of these, whether it is a metric or not.

10. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.
11. Suppose f is continuos on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
12. Evaluate $\lim_{x\to 0}(1/x) \int_0^x \sqrt{9 + t^2}dt$.

(**Bonus**) A real-valued function f defined in (a, b) is said to be **convex** if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Prove that every convex function is continuous.