Singularities of Ball Quotients

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Arithmetic ball quotients appear in algebraic geometry when studying moduli spaces of complex varieties via period maps. A broad goal of algebraic geometry is to understand varieties by studying the geometry of coarse moduli spaces. There has therefore been an interest in understanding the geometry of arithmetic ball quotients. Behrens has shown that for the toroidal compactification of an arithmetic ball quotient defined by general imaginary quadratic number fields, if the dimension is at least 13, then all singularities are canonical. Maeda has applied this result to show that many ball quotients are of general type. These results do not apply to ball quotients defined by the Eisenstein integers. Several Hodge theoretic constructions of coarse moduli spaces of varieties lead to considering arithmetic ball quotients defined by lattices over the Eisenstein integers. For instance such constructions exist for the moduli space of smooth cubic surfaces and the moduli space of smooth cubic threefolds.

In this thesis, we apply Behrens' methods for studying singularities of ball quotients to ball quotients defined by the Eisenstein integers. Rather than obtaining a dimension bound guaranteeing canonical singularities, we instead give a classification of the types of noncanonical singularities that can arise on ball quotients defined by the Eisenstein integers, in terms of related cyclic quotients. Motivated by general type results, we use this classification to prove a local criterion for when pluricanonical forms will extend to a resolution of singularities over these noncanonical singularities. We also consider singularities on another type of locally symmetric variety: generalized ball quotients of type $(2, n)$ defined by the Eisenstein integers.

Dedication

To my teachers.

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Introduction

The study of moduli spaces is a central topic in modern algebraic geometry. In this thesis, we will study the geometry of arithmetic ball quotients, which appear as higher dimensional analogues of the classical construction of modular curves as moduli spaces of elliptic curves (see [vG92] for a modular interpretation of ball quotients). To be more precise, we will study singularities on ball quotients with the goal of understanding the Kodaira dimension of ball quotients.

An important and motivating example of a moduli space arises when studying isomorphism classes of complex tori. We recall the details of this construction. A lattice in $\mathbb{C}$ is a free $\mathbb{Z}$-module Λ of rank 2 embedded in $\mathbb{C}$ such that for any two generators $\tau_1, \tau_2 \in \Lambda$, $\tau_1/\tau_2 \notin \mathbb{R}$. A 1-dimensional complex torus is given by $X = \mathbb{C}/\Lambda$ where Λ is a lattice in $\mathbb{C}$. The complex torus comes equipped with a natural $\mathbb{C}$-manifold structure given by defining charts via the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ (see [Mir95, p. 9] for details). Each 1-dimensional complex torus is a compact Riemann surface and, therefore, also carries the structure of a complex projective algebraic curve.

When are two complex tori isomorphic as $\mathbb{C}$-manifolds? To answer this question, we first note that, up to isomorphism, each complex torus can be described by an element of the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Indeed, two complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic if and only if $\Lambda = \gamma\Lambda'$ where $\gamma \in \mathbb{C}$, in this case we say Λ and Λ' are homothetic (see [Mir95, Proposition 1.11, p. 63]). Therefore, if τ_1 and τ_2 are generators for Λ, we define $\Lambda' = \tau_1^{-1}\Lambda$ so that 1 and τ_2/τ_1 are generators for Λ'. If $\mathrm{Im}(\tau_2/\tau_1) > 0$, then $\tau = \tau_2/\tau_1 \in \mathbb{H}$ is an element associated to $\mathbb{C}/\Lambda$. Otherwise, $-\tau_2/\tau_1 \in \mathbb{H}$ and is a generator for Λ', so is an element associated to $\mathbb{C}/\Lambda$.

So we have reduced the problem to considering lattices generated by 1 and $\tau \in \mathbb{H}$. By a classical result (see, e.g., [Mir95, Proposition 1.13, p. 64]), two elements $\tau, \tau' \in \mathbb{H}$ generate homothetic lattices if and only if there exists $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $A\tau = \tau'$, where $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Therefore, we may view the orbit space $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ as the moduli space of complex tori up to isomorphism, the points of this orbit space correspond to isomorphism classes of complex tori.

We may generalize this construction by instead taking a principal congruence subgroup $\Gamma(N) \subseteq \mathrm{SL}(2,\mathbb{Z})$, where

$$\Gamma(N) = \{A \in \mathrm{SL}(2,\mathbb{Z}) : A \equiv I \pmod{N}\}.$$

By convention, $\Gamma(0) = \mathrm{SL}(2,\mathbb{Z})$. The orbit space $\mathbb{H}/\Gamma(N)$ can be understood as a moduli space for complex tori along with information about the N-torsion points (see [Dia16, p. 38]).

In fact, each $\mathbb{H}/\Gamma(N)$ can be compactified and given the structure of a compact Riemann surface $X(N)$ (details can be found in [Dia16, Chapter 2])! One approach to studying complex tori is to study the geometry of these moduli spaces. As $X(N)$ is a compact Riemann surface, one of the first invariants we should try to determine is its genus. As we will see shortly, determining the Kodaira dimension of arithmetic ball quotients can be viewed as a higher dimensional analogue of determining the genus of a modular curve.

1.1 Background and Motivation

This thesis will be mainly concerned with the types of singularities that can appear on arithmetic ball quotients defined by the Eisenstein integers. The goal of this section is to provide the reader with the necessary background on Kodaira dimension to appreciate how these results about singularities can be useful for proving general type results.

1.1.1 Background on Kodaira dimension

In what follows, all varieties are assumed to be defined over the complex numbers. Let X be a smooth projective variety (i.e., a smooth compact complex manifold that can be embedded in complex projective space as the zero set of some algebraic equations) and $\mathcal{L}$ a holomorphic line bundle on X. For each $m \geq 1$, if the space of global sections of the m-th tensor power of the line bundle is non-trivial, i.e., $H^0(X, \mathcal{L}^{\otimes m}) \neq 0$, let $\varphi_m : X \dashrightarrow \mathbb{P}^N$ be the rational map induced by a choice of basis of the global sections $\mathcal{L}^{\otimes m}$. The Iitaka dimension of $\mathcal{L}$ is defined to be

$$\kappa(X, \mathcal{L}) = \max_{m \geq 1} \dim \varphi_m(X)$$

If $H^0(X, \mathcal{L}^{\otimes m}) = 0$ for all $m \geq 1$, then we say $\kappa(X, \mathcal{L}) = -\infty$. A line bundle with maximal Iitaka dimension is called big.

Every nonsingular projective variety has a canonical line bundle given by the top wedge of the holomorphic cotangent bundle $\omega_X = \bigwedge^{\dim X} \Omega^1_X$. The Kodaira dimension of X is defined to be the Iitaka dimension of the canonical bundle,

$$\kappa(X) = \kappa(X, \omega_X).$$

Recall, when $\kappa(X) = \dim(X)$, we say that X is of general type.

The Kodaira dimension gives a coarse invariant for varieties up to birational equivalence. In the case of curves, we obtain the following classification due to Kodaira dimension:

Genus	$g = 0$	$g = 1$	$g \geq 2$
$\kappa(X)$	$-\infty$	0	1

1.1.2 Singularities and Kodaira dimension

Our eventual goal is to determine the Kodaira dimension of various (coarse) moduli spaces parameterizing various varieties of different types. Such spaces are not typically smooth projective varieties, so we extend our definition of Kodaira dimension to a wider class of varieties. Let X be a normal quasi-projective variety. Let $\overline{X}$ be the Zariski closure of X and let $\widetilde{X} \to \overline{X}$ be a resolution of singularities. The Kodaira dimension of X is defined to be the Kodaira dimension of $\widetilde{X}$. This is independent of the choice of resolution of singularities $\widetilde{X}$ of X.

For a normal quasi-projective variety, there is a notion of a canonical sheaf $\omega_X = \iota_* \omega_{X^\circ}$, where X° is the smooth locus, which extends the notion of the canonical bundle to normal quasi-projective varieties. We often write this in additive notation as K_X, i.e., the Weil divisor associated to ω_X. While ω_X need not correspond to a line bundle on X, we say X is $\mathbb{Q}$-Gorenstein if rK_X is a Cartier divisor for some r, in other words, rK_X corresponds to a line bundle on X. It is natural to ask how the Iitaka dimension of rK_X relates to the Kodaira dimension of X (i.e., the Iitaka dimension of $K_{\widetilde{X}}$). Let $f : \widetilde{X} \to X$ be a resolution of singularities with exceptional divisors $\{E_i\}$. Then we have

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum_i a_i E_i,$$

If $a_i \geq 0$ for all i, then every section of rK_X pulls back to a section of $rK_{\widetilde{X}}$ so that $\kappa(X) \geq \kappa(X, \omega_X^{\otimes r})$. In this case we say that X has canonical singularities.

The upshot from the discussion above is that if the canonical sheaf on X is big (has maximal Iitaka dimension) and X has canonical singularities, then X is of general type. So the basic strategy to showing that X is of general type is to show that X has canonical singularities and that ω_X is big.

1.1.3 Kodaira dimension of moduli spaces

A coarse moduli space is a parameter space for some fixed class of variety. One of the broad goals of moduli theory is to understand the geometry of these spaces. Part of the beauty of this subject is that these moduli of varieties are in fact varieties themselves! As a first step in studying the geometry of moduli spaces, we can try to compute the Kodaira dimension of moduli spaces.

Apart from classification, why would one be interested in the Kodaira dimension of a moduli space? Heuristically, a moduli space having small Kodaira dimension means that you can write equations in projective space to represent most objects you are parameterizing. A moduli space having large Kodaira dimension (being of general type) means you cannot write such equations.

An example to keep in mind is the moduli space of nonsingular genus 3 curves M_3. The space M_3 has negative Kodaira dimension and a general genus 3 curve is given by a quartic equation in $\mathbb{P}^2$.

Table 1.1 gives a summary of general type results for some well known moduli spaces, namely $\overline{\mathcal{A}_g}$, the moduli of principally polarized abelian varieties, $\overline{M_g}$, the moduli of smooth curves of genus g, and $\overline{\mathcal{F}^0_{(2d)}}$, the moduli of $K3$ surfaces with degree $2d$ polarization. A theme from these results is that when the size of the invariants is small, we expect the Kodaira dimension to be small and when the size of the invariants is large, we expect the space to be of general type.

Moduli space	Is general type for...
$\overline{\mathcal{A}_g}$	$g \geq 7$ [Tai82] [Mum83]
$\overline{M_g}$	$g \geq 24$ [HM82][Har84][EH87], $g = 22, 23$ [FJP25]
$\overline{\mathcal{F}^0_{(2d)}}$	$d > 61, d = 46, 50, 54, 57, 58, 60$ [GHS07]

Table 1.1: General Type Results

In all of these examples, the proof involves showing that the canonical sheaf on M is big. As previously discussed, showing K_M is big implies M is of general type **if** M has at worst canonical singularities. In the case of $\overline{\mathcal{A}_g}$, Tai shows that the singularities are canonical for $g \geq 5$. Similarly Gritsenko, Hulek, and Sankaran show that $\overline{\mathcal{F}^0_{(2d)}}$ has canonical singularities. In contrast, $\overline{M}_g$ does **not** have canonical singularities. Therefore, by definition not every section of the canonical bundle extends to a section on a resolution of singularities. Despite this, Harris and Mumford were still able to show that **enough** sections extend to a resolution in order to ensure that $\overline{M_g}$ is of general type.

1.1.4 Ball Quotients

We will be studying arithmetic ball quotients, which we construct as follows. Let K be an imaginary quadratic number field and let $\mathcal{O}$ denote the ring of integers of K. Let Λ be a Hermitian $\mathcal{O}$-lattice of signature $(1, n)$, i.e., a free $(n+1)$-rank $\mathcal{O}$-module with Hermitian pairing h that extends to be a Hermitian pairing of signature $(1, n)$ on the complexification (see Definition 2.2.1 for all details). We define a complex n-ball by first extending scalars to obtain an $(n+1)$-dimensional $\mathbb{C}$-vector space $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathcal{O}} \mathbb{C}$, then considering the ($\mathbb{C}$-analytic) open set of $\mathbb{P}(\Lambda_{\mathbb{C}})$,

$$\mathbb{B}^n_\Lambda = \{[z] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : h(z, z) > 0\}.$$

To define a quotient of $\mathbb{B}^n_\Lambda$ with nice properties, we let Γ be a finite index subgroup of the unitary group $\mathrm{U}(\Lambda)$. The orbit space $\mathbb{B}^n_\Lambda/\Gamma$ is an arithmetic ball quotient defined by $\mathcal{O}$ (or defined by K, we will use these terms interchangeably). An arithmetic ball quotient defined by the Eisenstein integers is given when we let $K = \mathbb{Q}\left(\sqrt{-3}\right) = \mathbb{Q}(\zeta_3)$, where ζ_3 is a primitive third root of unity.

Arithmetic ball quotients are locally symmetric varieties and are, therefore, quasi-projective $\mathbb{C}$-varieties by the general theory of [BB66]. Because of this, arithmetic ball quotients have a unique minimal compactification as a projective variety called the Satake–Baily–Borel compactification and denoted $(\mathbb{B}^n_\Lambda/\Gamma)^*$ (see [Mil05, Theorem 3.12, Corollary 3.16, p. 291-293] for an overview). Similar to the case of the Satake–Baily–Borel compactification of the moduli space of principally polarized abelian varieties of dimension ≥ 2, the boundary $(\mathbb{B}^n_\Lambda/\Gamma)^* - \mathbb{B}^n_\Lambda/\Gamma$ has codimension ≥ 2. Due to the general theory of [AMRT10], there is a unique toroidal compactification, denoted $\overline{\mathbb{B}^n_\Lambda/\Gamma}$, such that the boundary $\overline{\mathbb{B}^n_\Lambda/\Gamma} - \mathbb{B}^n_\Lambda/\Gamma$ is a divisor.

In general, Γ may have torsion leading to singularities on $\overline{\mathbb{B}^n_\Lambda/\Gamma}$. Although one can always find a finite index subgroup $\Gamma' \subseteq \Gamma$ such that Γ' is neat (see [Mil05, Proposition 3.5, p. 288]), in the context of studying the Kodaira dimension of ball quotients as moduli spaces, this does not allow us to avoid the possible issue of singularities. It is classically known for locally symmetric varieties that for any $z \in \mathbb{B}^n_\Lambda$, the stabilizer $G = \mathrm{stab}_\Gamma(z)$ is finite (details given in Lemma 2.1.2). Therefore, locally near $\overline{z} \in \mathbb{B}^n_\Lambda/\Gamma$, $\mathbb{B}^n_\Lambda/\Gamma \cong \mathbb{B}^n_\Lambda/G$. We will be interested in the finite cyclic quotients locally at $\overline{z}$ via $\mathbb{B}^n_\Lambda/\langle g\rangle$ for $g \in G$. The reason we consider such quotients is due to the Reid–Tai criterion and [Tai82, Proposition 3.1] (for further explanation see Section 4.1 and Section 6.1).

Due to the relationship between canonical singularities and Kodaira dimension, we will study the singularities appearing on arithmetic ball quotients. Niko Behrens showed that for toroidal compactifications of arithmetic ball quotients $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ defined over imaginary quadratic number fields $\mathbb{Q}(\sqrt{-D})$ with $D > 3$, if $n \geq 13$, then $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ has at worst canonical singularities [Beh11, Theorem 4]. In this thesis, we consider the case of arithmetic ball quotients defined by $\mathbb{Q}(\sqrt{-3})$.

We briefly remark that several examples of arithmetic ball quotients arising from studying moduli spaces via period maps are given by lattices defined over $\mathbb{Q}(\sqrt{-3})$. In particular, both the ball quotient models for the moduli space of smooth cubic surfaces [ACT00] and the moduli space of smooth cubic threefolds [ACT06] have descriptions as Zariski open sets of ball quotients defined by the Eisenstein integers.

1.2 Main Results

In this thesis we prove three main results about singularities on ball quotients defined by the Eisenstein integers. We prove the following theorem giving a classification of all noncanonical quotient singularities, in terms of cyclic quotients.

Theorem 1.2.1 (Theorem 5.2.7)**.** *If $n \geq 10$ and $\mathbb{B}^n_\Lambda$ is defined by the Eisenstein integers, then the singularities of the toroidal compactification $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ are either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g\rangle \subseteq G$ so that $\mathbb{C}^n/\langle g\rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, $\frac{1}{12}(1,2,7)$, or $\frac{1}{12}(1,1,2,7)$.*

Corollary 1.2.2. *If there are no stabilizers with order divisible by 2 or 3, then the singularities of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ are canonical.* □

The benefit of the classification of the singularities in the main result above is that for these types of cyclic quotient singularities, we can use methods from toric geometry to give a criterion for when we can guarantee certain pluricanonical forms extend to a resolution of singularities. As motivation for future work toward general type results, we prove that on the interior, $\mathbb{B}^n_\Lambda/\Gamma$, this criterion is given locally by vanishing conditions along divisors determined by lattice elements. We expect a similar result will hold at the boundary as well.

Proposition 1.2.3 (Proposition 6.2.8)**.** *Suppose $\eta \in H^0(\mathbb{B}^n_\Lambda, kK_{\mathbb{B}^n_\Lambda})$ defines a form $\eta \in H^0(\overline{\mathbb{B}^n_\Lambda/\Gamma}, kK_{\overline{\mathbb{B}^n_\Lambda/\Gamma}})$, where $\mathbb{B}^n_\Lambda$ and Γ are defined by the Eisenstein integers and $n \geq 9$. Let $\overline{[\omega]} \in \mathbb{B}^n_\Lambda/\Gamma$ be a noncanonical singularity on the interior of the ball quotient. There exists a finite set $\mathcal{D}$ of $\mathcal{O}$-hyperplane divisors such that if η vanishes to order at least $6k$ along each $D \in \mathcal{D}$, then η extends to a regular pluricanonical form on a resolution of singularities of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ near $\overline{[\omega]}$.*

We also consider generalized ball quotients of type $(2, n)$ defined by a lattice over the Eisenstein integers and prove a similar result to Theorem 5.2.7 for the interior of such quotients.

Theorem 1.2.4 (Theorem 7.2.6)**.** *If $n \geq 6$, then the singularities of $\mathbb{B}_\Lambda(2,n)/\Gamma$ are either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^{2n}/G$, where the finite group G admits a cyclic subgroup $\langle g \rangle \subseteq G$ so that $\mathbb{C}^{2n}/\langle g \rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, or $\frac{1}{6}(1,1,1,1)$. When $n = 2$, there may also be singularities of type $\frac{1}{6}(1,1,2)$. When $n = 3$, there may also be singularities of type $\frac{1}{6}(1,1,1)$ or $\frac{1}{6}(1,1,1,2)$. When $n = 5$, there may also be singularities of type $\frac{1}{6}(1,1,1,1,1)$.*

1.3 Organization

Chapter 2 defines the spaces we are studying. In particular, we define what it means for a ball quotient to be defined 'by the Eisenstein integers'.

Chapter 3 studies the representation theory of finite cyclic groups over imaginary quadratic number fields. Behrens' result is recalled and a result is proven for representations over prime cyclotomic fields.

Chapter 4 contains the technical heart of the thesis. In this chapter we use the representation theory results to study cyclic quotient singularities appearing on the interior of ball quotients defined by the Eisenstein integers. We show how the methods for studying singularities on the moduli space of K3 surfaces in [GHS07] and singularities on ball quotients defined by general imaginary quadratic number fields in [Beh11] do not lead to a dimension bound guaranteeing canonical singularities for ball quotients defined by the Eisenstein integers. Rather, these methods allow us to give a classification of possible noncanonical singularities arising at the level of cyclic quotient singularities. We first study elements of $\mathbb{B}^n_\Lambda$ that are not fixed by quasi-reflections in Section 4.3 and then study elements fixed by quasi-reflections in Section 4.4.

Chapter 5 addresses the singularities that can appear on the boundary of the toroidal compactification of $\mathbb{B}^n_\Lambda/\Gamma$. The first main result of the thesis is found at the end of this chapter and is a summary of all of the results from Chapters 4 and 5.

Chapter 6 lays the foundation for proving general type results for ball quotients defined by the Eisenstein integers. In particular, we prove the second main result of this thesis about a local criterion for extending pluricanonical forms over the noncanonical singularities that can appear on ball quotients defined by the Eisenstein integers.

Chapter 7 considers the singularities arising on another type of locally symmetric variety. We prove similar results to those found in Chapter 4 for generalized ball quotients of type $(2, n)$ defined by the Eisenstein integers.

Appendix A gives background on results related to the Reid–Tai criterion about extension of pluricanonical forms over finite cyclic singularities. These results are known in the literature, we collect them in one place for convenience.

Appendix B provides the code used for computer computation throughout the thesis.

Chapter 2

Arithmetic Ball Quotients

In this chapter we define arithmetic ball quotients through the lens of locally symmetric varieties. We will then consider the case of ball quotients given by lattices defined over the ring of integers $\mathcal{O}$ of an imaginary quadratic number field K. We are particularly interested in the case where $\mathcal{O} = \mathbb{Z}[\zeta_3]$.

2.1 Ball Quotients in Full Generality

Let V be an $(n+1)$-dimensional $\mathbb{C}$-vector space with Hermitian form $h : V \times V \to \mathbb{C}$ of signature $(1, n)$. By the Spectral Theorem,[1] a basis $\{e_0, \ldots, e_n\}$ can be chosen for V so that h is given by

$$h(z, w) = \overline{z_0}w_0 - \overline{z_1}w_1 - \cdots - \overline{z_n}w_n.$$

In other words, $h(e_i, e_j) = 0$ for $i \neq j$, $h(e_0, e_0) = 1$, and $h(e_i, e_i) = -1$ for $i \neq 0$. We then define the n-dimensional ball as

$$\mathbb{B}^n = \{[z] \in \mathbb{P}(V) : h(z, z) > 0\}.$$

By taking the affine chart $z_0 \neq 0$ on $\mathbb{P}(V)$, we can view $\mathbb{B}^n$ as isomorphic to the usual n-dimensional complex ball. More precisely,

$$\mathbb{B}^n = \{[1 : z_1 : \ldots : z_n] \in \mathbb{P}(V) : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

[1] See [Axl24, Chapter 7] for a review of self-adjoint operators and the Spectral Theorem. Note that by choosing any basis $\{x_1, \ldots, x_{n+1}\}$ for V, the matrix $H = (h(x_i, x_j))_{i,j}$ is self-adjoint.

It can be shown that $\mathbb{B}^n$ has a canonical Hermitian metric with negative curvature [Mil05, Theorem 1.3], therefore $\mathbb{B}^n$ may be thought of as complex hyperbolic space. In fact, the group of holomorphic isometries of $\mathbb{B}^n$ acts transitively on $\mathbb{B}^n$ and there exists an involution with a unique fixed point [Hel78, Chapter VIII]. By definition, $\mathbb{B}^n$ is a Hermitian symmetric domain (see [Mil05, Chapter 1] for an overview of Hermitian symmetric domains).

Let

$$\mathrm{U}(1,n) = \{\varphi \in \mathrm{Aut}_{\mathbb{C}}(V) : \forall z, w \in V, h(\varphi(z), \varphi(w)) = h(z,w)\}.$$

We now show how $\mathrm{U}(1,n)$ can be viewed as a $\mathbb{R}$-algebraic group (see [Bor91, Chapter 1] for background and definitions related to algebraic groups). With respect to the basis $\{e_0, \dots, e_n\}$, we can realize $\mathrm{U}(1,n)$ as the complex matrix group:

$$\mathrm{U}(1,n) = \left\{A \in \mathrm{GL}_{n+1}(\mathbb{C}) : h(Az, Aw) = h(z,w), \forall z, w \in \mathbb{C}^{n+1}\right\}.$$

By checking the above condition for each pairing given on the basis, we obtain a finite list of conditions for the complex coordinates of A. Complex conjugation appears in the statement of these conditions. Therefore these conditions do not give $\mathrm{U}(1,n)$ as a complex subvariety of $\mathrm{GL}_{n+1}(\mathbb{C})$. Instead, we use the natural embedding of $\mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{GL}_{2(n+1)}(\mathbb{R})$ via

$$A = B + iC \mapsto \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

and note that the conditions on complex coordinates impose quadratic conditions on real coordinates. In other words, $\mathrm{U}(1,n)$ is a real subvariety of $\mathrm{GL}_{2n+2}(\mathbb{R})$.

The real algebraic group $\mathrm{U}(1,n)$ acts on $\mathbb{B}^n$ via holomorphic isometries. Indeed, $A \in \mathrm{U}(1,n)$ is compatible with h by definition and $h(z,z) > 0 \Rightarrow h(Az, Az) > 0$. We can show that all points $z \in \mathbb{B}^n$ can be mapped to the point $[1 : 0 : \cdots : 0] \in \mathbb{B}^n$ via an element of $\mathrm{U}(1,n)$. Since $\mathrm{U}(1,n)$ acts transitively on $\mathbb{B}^n$ and $\mathrm{U}(1) \times \mathrm{U}(n) \subseteq \mathrm{U}(1,n)$ is the stabilizer of the point $[1 : 0 : \cdots : 0]$, as smooth manifolds $\mathbb{B}^n \cong \mathrm{U}(1,n)/(\mathrm{U}(1) \times \mathrm{U}(n))$ [Mil05, Lemma 1.5, p. 272].

We will consider the space of orbits on $\mathbb{B}^n$ under the action of discrete subgroups $\Gamma \subseteq \mathrm{U}(1,n)$. In order for the space $\mathbb{B}^n/\Gamma$ to be an algebraic variety, we must impose the following technical condition on the subgroup Γ. Let $\mathrm{U}(1,n)(\mathbb{Q})$ denote the rational points of the real algebraic group $\mathrm{U}(1,n)$. Recall that two subgroups $H, K \subseteq G$ are commensurable if the index $[H : H \cap K] < \infty$ and $[K : H \cap K] < \infty$.

Definition 2.1.1. A subgroup $\Gamma \subseteq \mathrm{U}(1,n)(\mathbb{Q})$ is arithmetic if Γ is commensurable with $\mathrm{U}(1,n)(\mathbb{Q}) \cap \mathrm{GL}_N(\mathbb{Z})$ with respect to some embedding $U(1,n) \hookrightarrow \mathrm{GL}_N$ [Mil05, p. 287].

In full generality, an arithmetic ball quotient is the orbit space $\mathbb{B}^n/\Gamma$ where Γ is an arithmetic subgroup of $\mathrm{U}(1,n)$. From the perspective of Shimura varieties, ball quotients are denoted as left quotients $\Gamma\backslash\mathbb{B}^n$ because $\mathbb{B}^n \cong \mathrm{U}(1,n)/(\mathrm{U}(1) \times \mathrm{U}(n))$ is a quotient itself. In this thesis, we choose to denote ball quotients as right quotients $\mathbb{B}^n/\Gamma$ as we will not be explicitly using the framework of Shimura varieties.

The following lemma is well known for locally symmetric varieties and will be used throughout this thesis.

Lemma 2.1.2. *The stabilizer in Γ of any point $z \in \mathbb{B}^n$ is finite.*

Proof. This follows from [Shi94, Proposition 1.6(2)]. The proposition directly implies that if G is a locally compact topological group, $K \subseteq G$ is a compact subgroup, and $\Gamma \subseteq G$ is a discrete subgroup, then for all $z \in G/K$, the stabilizer $\mathrm{Stab}_\Gamma(z) = \{g \in \Gamma : g \cdot z = z\}$ is a finite set.

We apply this proposition to $G = \mathrm{U}(1,n)$ and $K = \mathrm{U}(1) \times \mathrm{U}(n)$, using the compactness of K. Recall that an arithmetic group Γ is discrete by definition. □

2.2 Ball Quotients Defined by Imaginary Quadratic Number Fields

In order to study arithmetic ball quotients, we use the following method for constructing arithmetic groups acting on $\mathbb{B}^n$. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic number field and let $\mathcal{O}$ be the ring of integers of K.

Definition 2.2.1. An $\mathcal{O}$-lattice of signature $(1,n)$ is a free $\mathcal{O}$-module Λ of rank $n+1$ with a

Hermitian form h of signature $(1,n)$; i.e., $h : \Lambda \times \Lambda \to \mathcal{O}$ is sesquilinear over $\mathcal{O}$,[2] Hermitian symmetric,[3] and the complexification of h is a Hermitian form on $\Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathcal{O}} \mathbb{C}$ such that there exists a basis so that with respect to this basis, the Hermitian form is given by the diagonal matrix $\mathrm{diag}(1, \underbrace{-1, \cdots, -1}_{n})$.

Given Λ, an $\mathcal{O}$-lattice of signature $(1,n)$, we define the unitary group of Λ to be

$$\mathrm{U}(\Lambda) = \{\varphi \in \mathrm{Aut}_{\mathcal{O}}(\Lambda) : \forall x \in \Lambda, h(\varphi(x), \varphi(x)) = h(x,x)\}.$$

Note that from Λ we can construct an $(n+1)$-dimensional $\mathbb{C}$-vector space $\Lambda_{\mathbb{C}}$ with a Hermitian form $h_{\mathbb{C}} : \Lambda_{\mathbb{C}} \times \Lambda_{\mathbb{C}} \to \mathbb{C}$ of signature $(1,n)$ given by extending h. We then use $\Lambda_{\mathbb{C}}$ and $h_{\mathbb{C}}$ as the input data to define the ball $\mathbb{B}^n_{\Lambda}$. Clearly $\mathbb{B}^n_{\Lambda} = \mathbb{B}^n$. We use the subscript Λ to denote that $\mathrm{U}(\Lambda)$ has a natural action on $\mathbb{B}^n_{\Lambda}$. If $\Gamma \subseteq \mathrm{U}(\Lambda)$ is any finite index subgroup, then Γ is an arithmetic subgroup of $\mathrm{U}(1,n)$. Indeed, $\mathrm{U}(\Lambda) = \mathrm{U}(1,n)(\mathbb{Q}) \cap \mathrm{GL}_{2n+2}(\mathbb{Z})$ with respect to the embedding $\mathrm{U}(1,n) \to \mathrm{GL}_{2n+2}$ given by an isomorphism $\mathrm{U}(\Lambda)(\mathbb{R}) \cong \mathrm{U}(1,n)$.

[2] By sesquilinear, we mean that for the nontrivial element $\alpha \in \mathrm{Gal}(\mathbb{Q}(\sqrt{-D}) : \mathbb{Q})$, $h(\lambda v, w) = \lambda h(v,w)$ and $h(v, \lambda w) = \alpha(w) h(v,w)$.

[3] By Hermitian symmetric, we mean $h(v,w) = \alpha(h(w,v))$ where α is the nontrivial element in $\mathrm{Gal}(\mathbb{Q}(\sqrt{-D}) : \mathbb{Q})$.

Chapter 3

Representation Theory

In this chapter, we study representations of finite cyclic groups over $\mathbb{Q}(\zeta_p)$-vector spaces. We begin by recalling a result of Behrens. We then extend this result to prime cyclotomic fields. Our goal is to understand the eigenvalues of all irreducible, faithful representations of finite cyclic groups over prime cyclotomic fields. The motivation for this goal is that it will allow us to analyze the singularities of $\mathbb{B}^n_\Lambda/\Gamma$ using the Reid–Tai criterion (see Chapter 4 for details). We note that for the case of $\mathbb{Q}(\zeta_3)$, Behrens' result can be applied directly.

Although this thesis will focus on the case of $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, we include our result for prime cyclotomic fields to fix notation and for potential future applications to studying singularities in more general settings. For an example of such a space defined by $\mathbb{Z}[\zeta_5]$, see [Kon05].

3.1 Representation Theory Over Imaginary Quadratic Number Fields

Let $\zeta_d = e^{2\pi i/d}$ and recall that the dth cyclotomic polynomial, denoted φ_d, is the minimal polynomial of ζ_d over $\mathbb{Q}$. We will use $\phi(d)$ to denote Euler's totient function.

Proposition 3.1.1 ([Wei27]). *Write $d = 2^{a_0}p_1^{a_1}\cdots p_r^{a_r}$, i.e., d has $r+1$ prime factors with multiplicity a_i. If $a_0 = 0$ or $a_0 = 1$, then there are exactly $2^r - 1$ quadratic number fields in which φ_d is reducible, in this case the quadratic number fields $\mathbb{Q}(\sqrt{D})$ are given by taking D to be the product of any $r' \leq r$ distinct numbers $(-1)^{(p_i-1)/2}p_i$. If $a_0 = 2$, then there are exactly $2^{r+1} - 1$ quadratic number fields in which φ_d is reducible, in this case $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{\pm D})$ where D is given by the same products as before. If $a_0 \geq 3$, then there are exactly $2^{r+2} - 1$ quadratic number fields in which*

φ_d is reducible, in this case $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\pm 2})$, $\mathbb{Q}(\sqrt{\pm D})$, and $\mathbb{Q}(\sqrt{\pm 2D})$ where D is given by the same products as before.

Example 3.1.2. The cyclotomic polynomial φ_p for an odd prime p is only reducible over one quadratic number field: $\mathbb{Q}\left(\sqrt{(-1)^{(p-1)/2}p}\right)$. This is imaginary quadratic if and only if $p \equiv 3 \mod 4$.

Example 3.1.3. The cyclotomic polynomial φ_{30} is reducible over $2^2 - 1 = 3$ quadratic number fields. Since $30 = 2 \cdot 3 \cdot 5$, we have that the possible factors of D are -3 and 5. Therefore the three quadratic number fields over which φ_{30} is reducible are $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\sqrt{-15})$.

Example 3.1.4. The cyclotomic polynomial φ_{12} is reducible over $2^{1+1} - 1 = 3$ quadratic number fields. They are $\mathbb{Q}(i), \mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{-3})$.

Example 3.1.5. The cyclotomic polyomial φ_{120} is reducible over $2^{2+2} - 1 = 15$ quadratic number fields. They are $\mathbb{Q}(i), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{-10})$, $\mathbb{Q}(\sqrt{15}), \mathbb{Q}(\sqrt{-15}), \mathbb{Q}(\sqrt{30})$, and $\mathbb{Q}(\sqrt{-30})$.

Given a fixed d, we want to know how the roots of the cyclotomic polynomial φ_d split between irreducible factors over $\mathbb{Q}(\sqrt{D})$. When φ_d is irreducible over $\mathbb{Q}(\sqrt{D})$, all of the primitive dth roots of unity appear as roots of φ_d. Otherwise, we will see that φ_d has two irreducible factors whose roots are determined by the Kronecker symbol.

We say a is a quadratic residue mod p if there exists b such that $b^2 \equiv a \mod p$. Recall that the Legendre symbol is a function detecting the quadratic reciprocity of integers over odd primes p.

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \mod p, \\ -1, & \text{if } a \text{ is not a quadratic residue modulo } p, \\ 0, & \text{if } a \not\equiv 0 \mod p \end{cases}$$

The Kronecker symbol is an extension of the Legendre symbol for all integers n. If $n = up_1^{a_1} \cdots p_n^{a_n}$ is the unique prime factorization of n with $u \in \{1, -1\}$, then

$$\left(\frac{a}{n}\right) = \left(\frac{a}{u}\right) \prod_{j=1}^{n} \left(\frac{a}{p_j}\right)^{a_j},$$

where

$$\left(\frac{a}{-1}\right) = \begin{cases} 1, & a \geq 0 \\ -1, & a < 0 \end{cases}$$

and

$$\left(\frac{a}{2}\right) = \begin{cases} 0, & a \text{ even}, \\ 1, & a \equiv \pm 1 \mod 8 \cdot \\ -1, & a \equiv \pm 3 \mod 8 \end{cases}$$

The following is a restatement of [Beh11, Proposition 1] that classifies the irreducible, faithful representations of finite cyclic groups over imaginary quadratic number fields.

Proposition 3.1.6. *(i) If φ_d is irreducible over $\mathbb{Q}(\sqrt{-D})$, then up to isomorphism there is a unique irreducible, faithful representation of $\mathbb{Z}/d\mathbb{Z}$ over $\mathbb{Q}(\sqrt{-D})$. We denote this by $\rho : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(0)}\right)$, and one may take $V_d^{(0)} = \mathbb{Q}(\sqrt{-D})[x]/(\varphi_d(x))$ with $\rho(1)$ given by multiplication by x. We have $\dim V_d^{(0)} = \phi(d)$ and the eigenvalues of $\rho(1)$ are the primitive dth roots of unity.*

(ii) If ϕ_d is reducible over $\mathbb{Q}(\sqrt{-D})$, then up to isomorphism there are 2 irreducible, faithful representations of $\mathbb{Z}/d\mathbb{Z}$ over $\mathbb{Q}(\sqrt{-D})$. We denote this by $\rho^{(1)} : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(1)}\right)$ and $\rho^{(2)} : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(2)}\right)$, and one may take $V_d^{(1)} = \mathbb{Q}(\sqrt{-D})[x]/\left(\varphi_d^{(1)}(x)\right)$ and $V_d^{(2)} = \mathbb{Q}(\sqrt{-D})[x]/\left(\varphi_d^{(2)}(x)\right)$ with $\rho^{(1)}(1)$ and $\rho^{(2)}(1)$ given by multiplication by x, where $\varphi_d^{(1)}$ and $\varphi_d^{(2)}$ are the irreducible factors of φ_d over $\mathbb{Q}(\sqrt{-D})$. We have $\dim V_d^{(1)} = \dim V_d^{(2)} = \phi(d)/2$. The eigenvalues of $\rho^{(1)}(1)$ are ζ_d^a with $\left(\frac{a}{D}\right) = -1$ for $a \in (\mathbb{Z}/d\mathbb{Z})^*$. The eigenvalues of $\rho^{(2)}(1)$ are ζ_d^a for the remaining $a \in (\mathbb{Z}/d\mathbb{Z})^*$, i.e., the a with $\left(\frac{a}{D}\right) = 1$.

Remark 3.1.7. The naming convention here for $V_d^{(1)}$ and $V_d^{(2)}$ is different from [Beh11] in order to agree with Proposition 3.2.5 for prime cyclotomic fields.

Remark 3.1.8. A proof of this statement follows by applying Lemma 3.2.4 and examining how the roots split among irreducible factors over K in relation to Proposition 3.1.1.

Example 3.1.9. If $D = -3$, then first we have that φ_d is reducible over $\mathbb{Q}(\sqrt{D})$ if and only if $3 \mid d$ by Proposition 3.1.1. For each $a \in (\mathbb{Z}/d\mathbb{Z})^*$, we have that $\left(\frac{a}{-3}\right) = -1 \cdot \left(\frac{a}{3}\right)$, but a is a quadratic residue mod 3 if and only if $a \equiv 1 \mod 3$.

Example 3.1.10. Let $d = 21$ so that φ_d is reducible over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$. Note that each $V_{21}^{(k)}$ has dimension $\phi(21)/2 = 6$. Over $\mathbb{Q}(\sqrt{-3})$ we have that the eigenvalues of $V_{21}^{(1)}$ are $\zeta_{21}, \zeta_{21}^4, \zeta_{21}^{10}, \zeta_{21}^{13}, \zeta_{21}^{16}, \zeta_{21}^{19}$ and the eigenvalues of $V_{21}^{(2)}$ are $\zeta_{21}^2, \zeta_{21}^5, \zeta_{21}^8, \zeta_{21}^{11}, \zeta_{21}^{17}, \zeta_{21}^{20}$. Over $\mathbb{Q}(\sqrt{-7})$, we have that the eigenvalues of $V_{21}^{(1)}$ are $\zeta_{21}, \zeta_{21}^2, \zeta_{21}^4, \zeta_{21}^8, \zeta_{21}^{11}, \zeta_{21}^{16}$ and the eigenvalues of $V_{21}^{(2)}$ are $\zeta_{21}^5, \zeta_{21}^{10}, \zeta_{21}^{13}, \zeta_{21}^{17}, \zeta_{21}^{19}, \zeta_{21}^{20}$. Note that which eigenvalues appear in each irreducible representation depends on the number field.

3.2 Representation Theory Over Prime Cyclotomic Fields

We now prove a similar result for representations of cyclic groups over prime cyclotomic fields. While this is not necessary to study ball quotients defined by the Eisenstein integers, we include this result for future applications.

Lemma 3.2.1. *$\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_n)$ if and only if $p \mid n$.*

Proof. Suppose $n = kp$. Then ζ_n^k is a primitive pth root of unity. Therefore $\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_n)$. Suppose conversely that $p \nmid n$. Then $(n, p) = 1$. By [Was96, Proposition 2.4] or [Lan02, VI §3 Cor. 3.2, p.278], $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Thus, $\mathbb{Q}(\zeta_p) \not\subset \mathbb{Q}(\zeta_n)$. $\square$

We would like to study next the irreducible factors of ϕ_d over $\mathbb{Q}(\zeta_p)$ and how the primitive dth roots of unity distribute between factors. By the above lemma, we assume that

$$d = pk.$$

Note that $\mathbb{Q}(\zeta_d)/\mathbb{Q}$ is Galois, so $\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)$ is also Galois.

Lemma 3.2.2. *Suppose n divides d. The d-th roots of unity ζ_d^ℓ and $\zeta_d^{\ell'}$ are primitive if and only if $(\ell, d) = (\ell', d) = 1$, and in this case they are Galois conjugates over $\mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_n))$ if and only if $\ell \equiv \ell' \mod n$. This partitions the primitive d-th roots of unity into $\varphi(n)$ equivalence classes each of size $\varphi(d)/\varphi(n)$.*

Proof. Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_n))$. By [Dum04, Theorem 26, p. 596], $(\mathbb{Z}/d\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q})$ via $[k] \mapsto (\zeta_d \mapsto \zeta_d^k)$ and $(\mathbb{Z}/n\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n) : \mathbb{Q})$ via $[k] \mapsto (\zeta_n \mapsto \zeta_n^k)$. By the Fundamental Theorem of Galois Theory [Dum04, Theorem 14(4), p. 574], we obtain the short exact sequence

$$1 \to G \to \mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_n) : \mathbb{Q}) \to 1,$$

where the surjective morphism is given by sending $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q})$ to $\sigma|_{\mathbb{Q}(\zeta_n)}$. By applying the isomorphisms $(\mathbb{Z}/d\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q})$ via $[k] \mapsto (\zeta_d \mapsto \zeta_d^k)$ and $(\mathbb{Z}/n\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n) : \mathbb{Q})$ via $[k] \mapsto (\zeta_n \mapsto \zeta_n^k)$, we obtain

$$1 \to G \to (\mathbb{Z}/d\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^* \to 1,$$

where the surjective morphism is given by sending $[k]_d \in (\mathbb{Z}/d\mathbb{Z})^*$ to $[k]_n \in (\mathbb{Z}/n\mathbb{Z})^*$. Therefore, under these isomorphisms,

$$G = \{[k]_d \in (\mathbb{Z}/d\mathbb{Z})^* : k \equiv 1 \pmod n\}.$$

Suppose ζ_d^ℓ and $\zeta_d^{\ell'}$ are primitive dth roots of unity and are Galois conjugate over G via the element $\sigma = [k]_d \in G \subseteq (\mathbb{Z}/d\mathbb{Z})^*$, i.e., $\sigma(\zeta_d^\ell) = \zeta_d^{\ell k} = \zeta_d^{\ell'}$. Therefore $\ell k \equiv \ell' \pmod d \Rightarrow \ell \equiv \ell' \pmod n$ since $k \equiv 1 \pmod n$.

Conversely, suppose that $\ell \equiv \ell' \pmod n$ and $(\ell, d) = (\ell', d) = 1$. Then ζ_d^ℓ and $\zeta_d^{\ell'}$ are primitive d-th roots of unity. There exists k such that $\ell k \equiv \ell' \pmod d$ and $k \equiv 1 \pmod n$. Indeed, let $k \equiv \ell' \ell^{-1} \pmod d$. Note that $k \equiv 1 \pmod n$ since $\ell \equiv \ell' \pmod n \implies 1 \equiv \ell' \ell^{-1} \equiv j \pmod n$. Therefore, ζ_d^ℓ and $\zeta_d^{\ell'}$ are Galois conjugate over G via $[k]_d \in G \subseteq (\mathbb{Z}/d\mathbb{Z})^*$.

Let S denote the set of primitive dth roots of unity. The group G acts on S. The orbit $\mathrm{Orb}_G(\zeta_d^\ell)$ is the Galois conjugacy class containing ζ_d^ℓ. The stabilizer $\mathrm{Stab}_G(\zeta_d^\ell) = \{1\}$ since any automorphism of $\mathbb{Q}(\zeta_d)$ fixing a primitive dth root of unity must fix all dth roots of unity and is therefore trivial. By the Orbit-Stabilizer Theorem, the order of the G-orbit of ζ_d^ℓ is the index of stabilizer of ζ_d^ℓ in G, therefore $|\mathrm{Orb}_G(\zeta_d^\ell)| = |G|$. By the short exact sequence

$$1 \to G \to (\mathbb{Z}/d\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^* \to 1,$$

$|G| = |(\mathbb{Z}/d\mathbb{Z})^*|/|(\mathbb{Z}/n\mathbb{Z})^*| = \phi(d)/\phi(n)$. Each Galois conjugacy class is disjoint and $|S| = \phi(d)$, therefore this action partitions S into $\varphi(n)$ equivalence classes. □

The main result we need is that φ_{pk} factors over $\mathbb{Q}(\zeta_p)$ into $p-1$ irreducible factors, whose roots are the equivalence classes of Galois conjugate pk-th roots of unity. This follows almost immediately from results in Galois theory.

Lemma 3.2.3. *The cyclotomic polynomial $\phi_d(x)$ is reducible over $\mathbb{Q}(\zeta_p)$ if and only if $p \mid d$. In this case, $\varphi_d(x)$ factors into $p-1$ irreducible polynomials whose roots are given by the Galois conjugates of each primitive dth root of unity over* $\mathrm{Gal}(\mathbb{Q}(\zeta_d) : \mathbb{Q}(\zeta_p))$*.*

Proof. We have already shown that $\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_d)$ if and only if $p \mid d$. If $p \mid d$, then $\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_d)$. So the minimal polynomial of ζ_d over $\mathbb{Q}(\zeta_p)$ divides $\phi_d(x)$.

Conversely suppose that $\phi_d(x)$ is reducible over $\mathbb{Q}(\zeta_p)$. Since $\mathbb{Q}(\zeta_p)[x]$ is a UFD, there exists an irreducible factorization $\phi_d(x) = f_1(x)^{a_1} \cdots f_k(x)^{a_k}$ in $\mathbb{Q}(\zeta_p)[x]$. Since $\phi_d(x)$ has no repeated roots (the extension is separable), each $a_i = 1$ and $k > 1$. Let F_i be the field generated by the coefficients of $f_i(x)$. Clearly $f_i(x) \in F_i[x]$ and $F_i \subseteq \mathbb{Q}(\zeta_p)$ since $f_i(x) \in \mathbb{Q}(\zeta_p)[x]$. We claim that $F_i \subseteq \mathbb{Q}(\zeta_d)$. Indeed the roots of $f_i(x)$ are primitive dth roots of unity and so each coefficient of $f_i(x)$ can be expressed as products and sums of ζ_d. Thus each generator of F_i is contained in $\mathbb{Q}(\zeta_d)$, so $F_i \subseteq \mathbb{Q}(\zeta_d)$. Therefore $F = F_1 \cdots F_k$ is a subfield of $\mathbb{Q}(\zeta_d)$ and $\mathbb{Q}(\zeta_p)$, so $F \subseteq \mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_d)$. Notice that $F \neq \mathbb{Q}$ since otherwise we would have that each $F_i = \mathbb{Q}$ which implies that $\phi_d(x)$ is reducible over $\mathbb{Q}$, a contradiction. Thus $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_d) \neq \mathbb{Q}$. By [Was96, Proposition 2.4], we have that $(d, p) \neq 1 \Rightarrow p \mid d$ since p is prime.

The statement about how $\phi_d(x)$ factors is a consequence of Galois theory. Suppose $p \mid d$. Let ζ_d be a primitive dth root of unity and let

$$f(x) = \prod_{\substack{0 \leq \ell \leq d-1, \\ (\ell, d)=1, \\ \ell \equiv 1 \mod p}} \left(x - \zeta_d^{\ell}\right)$$

be the minimal polynomial of ζ_d over $\mathbb{Q}(\zeta_p)$. Note that we may express the minimal polynomial of ζ_d in this way by applying Lemma 3.2.2 since $\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_p)$ is Galois and the Galois group acts transitively on the roots of this polynomial. By choosing a representative for each of the $p-1$ equivalence classes of conjugate roots, we obtain the $p-1$ irreducible factors of $\phi_d(x)$. $\square$

Lemma 3.2.4. *Let K be a number field and let $f_1, \ldots, f_r$ be the irreducible factors of the cyclotomic polynomial φ_d over K. Up to isomorphism, there are r irreducible, faithful representations of $\mathbb{Z}/d\mathbb{Z}$ over K. They can be given explicitly by $\rho_k : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(K[x]/(f_k(x))\right)$ with $\varphi_k(1)$ given by multiplication by x. Moreover, $\dim \rho_k = \deg f_k$ and the eigenvalues of $\rho(1)$ are the roots of f_k.*

Proof. We will first consider the explicit representations ρ_k and show that they satisfy the properties given in the lemma. We will then show that up to isomorphism these can be the only irreducible, faithful representations of $\mathbb{Z}/d\mathbb{Z}$ over K.

Fix an irreducible factor f_k of φ_d. Then $V = K[x]/(f_k(x))$ is a K-vector space of dimension $\deg f_k$ and multiplication by x gives a K-linear automorphism of V. Let $\alpha : V \to V$ denote the multiplication by x automorphism, i.e., $\alpha(v) = x \cdot v$ where $v \in K[x]/(f_k(x))$. We claim that $\rho_k : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}(V)$ via $[1] \mapsto \alpha$ is an irreducible, faithful representation.

Fix the K-basis $\{1, x, x^2, \ldots, x^{\deg f_k - 1}\}$ for V. Let $f_k(x) = x^{\deg f_k} + a_{\deg f_k - 1} x^{\deg f_k - 1} + \cdots + a_1 x + a_0$. With respect to this basis, α is given by the matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{\deg f_k - 1} \end{pmatrix}.$$

Note that the characteristic polynomial of A is f_k. Moreover, the minimal polynomial of A is also f_k. Indeed, the minimal polynomial divides f_k and f_k divides some power of the minimal polynomial [Dum04, Proposition 20, p. 478], but the roots of f_k have multiplicity 1 because the roots of φ_d have multiplicity 1. Therefore the characteristic polynomial and minimal polynomial are equal. Since $f_k \mid x^d - 1$, we have that $A^d = I$. Therefore ρ_k is well-defined.

We now show that ρ_k is faithful. Suppose that $A^m = I$. Therefore $f_k(x) \mid x^m - 1$. The roots of $f_k(x)$ are primitive dth roots of unity, therefore $d \mid m$. Thus, $A^k \neq I$ for all $1 \leq k < d$ so ρ_k is injective.

We now show that ρ_k is irreducible. Suppose for a contradiction that ρ_k is reducible, i.e., A has a nontrivial invariant subspace W. Take any basis for W and extend it to a basis for $K[x]/(f_k(x))$. With respect to this basis we have that

$$\rho_k(1) = \left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right].$$

But then we have that the characteristic polynomial $f_k(x) = \det(B - \lambda I)\det(D - \lambda I)$, contradicting the irreducibility of $f_k(x)$.

Clearly $\dim \rho_k = \deg f_k$. The eigenvalues of $\rho_k(1)$ are precisely the roots of the characteristic polynomial of $\rho_k(1)$, which is $f_k(x)$. We now show that ρ_k and $\rho_{k'}$ are nonisomorphic representations. Suppose there exists an isomorphism $\psi : K[x]/(f_k(x)) \to K[x]/(f_{k'}(x))$ such that $\psi(\rho_k(1)v) = \rho_{k'}(1)\psi(v)$ for all $v \in K[x]/(f_k(x))$. Let λ be an eigenvalue of $\rho_k(1)$ with corresponding eigenvector v, then $\rho_{k'}(1)\psi(v) = \psi(\rho_k(1)v) = \psi(\lambda v) = \lambda\psi(v)$. Therefore, $\psi(v)$ is a λ-eigenvector of $\rho_{k'}(1)$. So $f_k(x)$ and $f_{k'}(x)$ share a root. But $f_k(x)$ and $f_{k'}(x)$ are irreducible factors of φ_d, which has roots of multiplicity 1, therefore $k = k'$.

We now show that the ρ_k we found are the only irreducible, faithful representations of $\mathbb{Z}/d\mathbb{Z}$ over K up to isomorphism. Let $\rho : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}(K^n)$ be an irreducible, faithful representation. Fix any basis for K^n and let A be the matrix for $\rho(1)$ with respect to this basis. Let $\chi_A(x)$ and $m_A(x)$ denote the characteristic polynomial and minimal polynomial for A, respectively.

Note that $A^d = I$, therefore $\chi_A(x) \mid x^d - 1$. By [Dum04, Proposition 20, p. 478], $m_A(x) \mid$

$\chi_A(x) \mid m_A(x)^\ell$. But $x^d - 1$ has distinct roots, therefore $\chi_A(x) = m_A(x)$.

We now show that $m_A(x)$ is irreducible over K. Suppose for a contradiction that $m_A(x)$ is reducible over K, i.e., $m_A(x) = f(x)g(x)$ for some monic $f(x), g(x) \in K[x]$ with degree at least 1. We then consider the subspace $W = f(A)(V)$. We have that $W \neq \{0\}$ since then $f(A) = 0 \Rightarrow m_A(x) \mid f(x)$, which is a contradiction. Also, $W \neq V$ since then $f(A)$ would be invertible, so $f(A)g(A) = 0 \Rightarrow g(A) = 0$, again contradicting the fact that $m_A(x)$ is the minimal polynomial. Moreover, W is A-invariant. Indeed if $w \in W$, then there exists $v \in V$ such that $f(A)v = w$, and

$$Aw = A(f(A)v) = (Af(A))v = f(A)(Av) \in W.$$

We have shown that $\rho|_W$ is a proper, nontrivial subrepresentation of ρ, contradicting the irreducibility of ρ.

We now show that $m_A(x) \mid \varphi_d(x)$. Recall that

$$x^d - 1 = \prod_{k|d} \varphi_k(x)$$

are the irreducible factors of $x^d - 1$ over $\mathbb{Q}$. Since $m_A(x)$ is irreducible over K, it must divide only one $\varphi_k(x)$ with $k \mid d$. Since $m_A(x) \mid \varphi_k(x) \mid x^k - 1$, we have that $A^k = I$. But ρ is faithful, so $A^k \neq I$ for all $1 \leq k < d$. Therefore, $m_A(x) \mid \varphi_d(x)$.

It follows that $m_A(x) = f_k(x)$, an irreducible factor of $\varphi_d(x)$ over K. By [Dum04, Theorem 14, p. 476], after changing basis, the Rational Canonical Form for $\rho(1)$ is given by the companion matrix for $f_k(x)$. Therefore by [Dum04, Theorem 15, p. 476], ρ is isomorphic to ρ_k. □

We then obtain the result for irreducible representations of finite groups over $\mathbb{Q}(\zeta_p)$ corresponding to [Beh11, Proposition 1].

Proposition 3.2.5. *(i) If φ_d is irreducible over $\mathbb{Q}(\zeta_p)$, then up to isomorphism there is a unique irreducible, faithful representation of $\mathbb{Z}/d\mathbb{Z}$ over $\mathbb{Q}(\zeta_p)$. We denote this by $\rho : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(0)}\right)$, and one may take $V_d^{(0)} = \mathbb{Q}(\zeta_p)[x]/(\varphi_d(x))$ with $\rho(1)$ given by multiplication* *by x. We have* $\dim V_d^{(0)} = \phi(d)$ *and the eigenvalues of* $\rho(1)$ *are the primitive* d*th roots of unity.*

(ii) If φ_d *is reducible over* $\mathbb{Q}(\zeta_p)$*, then up to isomorphism there are* $p-1$ *irreducible, faithful representations of* $\mathbb{Z}/d\mathbb{Z}$ *over* $\mathbb{Q}(\zeta_p)$*. We denote this by* $\rho^{(1)} : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(1)}\right)$*, ...,* $\rho^{(p-1)} : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}\left(V_d^{(p-1)}\right)$*, and one may take* $V_d^{(k)} = \mathbb{Q}(\zeta_p)[x]/\left(\varphi_d^{(k)}(x)\right)$ *with* $\rho^{(k)}(1)$ *given by multiplication by* x*, where* $\varphi_d^{(k)}$ *is the irreducible factor of* φ_d *over* $\mathbb{Q}(\zeta_p)$ *with roots* ζ_d^a *such that* $a \equiv k \mod p$*. We have* $\dim V_d^{(k)} = \phi(d)/(p-1)$*. The eigenvalues of* $\rho^{(k)}(1)$ *are the primitive* d*th roots of unity* ζ_d^a *such that* $a \equiv k \mod p$*.*

Proof. For the first statement, apply Lemma 3.2.4 directly.

For the second statement, apply Lemma 3.2.4 to the $p-1$ irreducible factors of φ_d given by Lemma 3.2.3. Then apply Lemma 3.2.2 and Lemma 3.2.4 to give the statement about the eigenvalues. □

3.3 Reducing Representations

The following lemma is elementary and will be used in later chapters to decompose reducible representations of cyclic groups into irreducible faithful representations of cyclic groups of varying orders.

Lemma 3.3.1. *Let* V *be a* k*-vector space and* d *be an integer such that* $d \in k^*$ *and let* $\rho : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}(V)$ *be a representation. Then* ρ *decomposes uniquely into irreducible, faithful representations* $\rho = \rho_1 \oplus \cdots \oplus \rho_m$ *with* $\rho_j : \mathbb{Z}/d_j\mathbb{Z} \to \mathrm{Aut}(V_j), V_j \subseteq V$ *and* $d_j \mid d$*.*

Proof. By Maschke's Theorem, we have a unique decomposition $\rho = \rho'_1 \oplus \cdots \rho'_m$ where each $\rho'_j : \mathbb{Z}/d\mathbb{Z} \to \mathrm{Aut}(V_j)$ is irreducible. To obtain irreducible faithful representations, we quotient the domain of each subrepresentation by its kernel, i.e., $\rho_j : (\mathbb{Z}/d\mathbb{Z})/\ker(\rho_j) \cong \mathbb{Z}/d_j\mathbb{Z} \to \mathrm{Aut}(V_j)$. Note that each ρ_j is clearly faithful since its kernel is now trivial. It only remains to show that each ρ_j is still irreducible. Suppose for a contradiction that $\rho_j = \mu \oplus \eta$ with μ and η nontrivial. Since $\ker(\rho'_j)$ acts trivially on V_j and hence on the subspaces defined by μ and η, we have that μ and η

each lift to nontrivial representations μ' and η' of ρ'_j so that $\rho'_j = \mu' \oplus \eta'$. But this contradicts the irreducibility of ρ'_j. $\square$

Chapter 4

Singularities on the Interior

In this chapter we study the singularities arising on the interior of ball quotients defined by $\mathbb{Z}[\zeta_3]$. Our main tools will be Proposition 3.2.5 and the Reid–Tai criterion. Our methods are similar to those used in [Beh11, Theorem 2] and [GHS07, Section 2]. In contrast to these other results, we will be unable to obtain a dimension bound that guarantees canonical singularities for $\mathbb{B}^n_\Lambda$. Instead, we give a complete classification of noncanonical cyclic singularities that can arise.

4.1 Reid–Tai Criterion

The Reid–Tai criterion allows us to determine whether a finite quotient singularity is a canonical singularity. For more details and a collection of related results, see Appendix A. The following statement of the Reid–Tai criterion will suffice for our purposes.

Let $A \in \mathrm{GL}(\mathbb{C}^k)$ be of order ℓ and fix $\zeta = e^{2\pi i/\ell}$. Let $\zeta^{a_1}, \ldots, \zeta^{a_k}$ denote the eigenvalues of A on $\mathbb{C}^k$ where $0 \leq a_i < \ell$. Recall that we say A is a quasi-reflection if there is a unique j such that $\zeta^{a_j} \neq 1$. The Reid–Tai sum of A is defined to be

$$\sum(A) := \sum_{i=1}^{k} \frac{a_i}{\ell}.$$

Theorem 4.1.1 (Reid–Tai criterion, [Beh11])**.** *Let H be a finite subgroup of $\mathrm{GL}(\mathbb{C}^k)$ without quasi-reflections. Then $\mathbb{C}^k/H$ has canonical singularities if and only if*

$$\sum(A) \geq 1$$

for every $A \in H$, $A \neq I$.

4.2 Linearizing the Action of G

Fix a point $[\omega] \in \mathbb{B}^n_\Lambda$ and let $G = \mathrm{Stab}_\Gamma([\omega])$. Recall that by Lemma 2.1.2, $|G| < \infty$. Let $\mathbb{W} = \mathrm{span}_\mathbb{C}(\omega)$ be the line in $\Lambda_\mathbb{C}$ corresponding to the point $[\omega] \in \mathbb{P}\Lambda_\mathbb{C}$.

There is an open neighborhood U of $[\omega]$ such that G acts on U and, locally near $\overline{[\omega]}$, $\mathbb{B}^n_\Lambda/\Gamma \cong U/G$. Moreover, G acts on the tangent space $T_{[\omega]}\mathbb{B}^n_\Lambda$ and, locally near the origin, $T_{\omega]}\mathbb{B}^n_\Lambda/G \cong U/G$.

Our goal is to apply the Reid–Tai criterion to the action of G on the tangent space $T_{[\omega]}\mathbb{B}^n_\Lambda$. Since $\mathbb{B}^n_\Lambda$ is a complex analytic open set in $\mathbb{P}\Lambda_\mathbb{C}$, the Euler sequence gives a canonical isomorphism $T_{[\omega]}\mathbb{B}^n_\Lambda = \mathrm{Hom}(\mathbb{W}, \Lambda_\mathbb{C}/\mathbb{W})$. We will use our result about irreducible faithful $\mathbb{Q}(\zeta_3)$-representations in order to determine the eigenvalues of some $g \in G$ acting on the tangent space at $[\omega]$.

Let $K = \mathbb{Q}(\zeta_3)$ and fix an embedding of K into $\mathbb{C}$. We have that G acts on $\Lambda_K = \Lambda \otimes_\mathcal{O} K$. Therefore for each $g \in G$ there is a representation $\rho_g : \langle g \rangle \to \mathrm{Aut}\,(\Lambda_K)$. By Lemma 3.3.1 and Proposition 3.2.5, we have the decomposition

$$\Lambda_K \cong \bigoplus_{j=1}^{m} V_{d_j}^{(k_j)}. \tag{4.2.1}$$

Moreover, Proposition 3.2.5 also gives the eigenvalues of g acting on $\Lambda_\mathbb{C}$ as these will arise from the eigenvalues of g acting on each irreducible component.

Since $[\omega] \in \mathbb{P}\Lambda_\mathbb{C}$ is fixed by g, we have that $\omega \in \Lambda_\mathbb{C}$ is an eigenvector under the action of g. Let λ denote the eigenvalue of g acting on $\mathbb{W}$. Let $r = \mathrm{ord}(\lambda)$. Since λ appears as an eigenvalue of g acting on $\Lambda_\mathbb{C}$, we must have that at least one $V_{d_j}^{(k_j)}$ is isomorphic to $V_r^{(k)}$. Note that this does not ensure that $\mathbb{W} \subset V_r^{(k)} \otimes_K \mathbb{C}$, in fact several $V_{d_j}^{(k_j)}$ may be isomorphic to $V_r^{(k)}$. All that can be guaranteed is that $\mathbb{W}$ is contained in the λ-eigenspace of g acting on $\Lambda_\mathbb{C}$. The upshot of the above discussion is that we can now determine the eigenvalues of g acting on $\mathrm{Hom}(\mathbb{W}, \Lambda_\mathbb{C}/\mathbb{W})$.

Lemma 4.2.1. *Let $[\omega] \in \mathbb{B}^n_\Lambda$ and $g \in \mathrm{Stab}_{\mathrm{U}(\Lambda)}([\omega])$. Let $\lambda, \lambda_1, \ldots, \lambda_n$ be the eigenvalues of g acting on $\Lambda_\mathbb{C}$, where $g \cdot \omega = \lambda\omega$. The eigenvalues of g acting on $T_{[\omega]}\mathbb{B}^n_\Lambda$ are given by $\lambda^{-1}\lambda_1, \ldots, \lambda^{-1}\lambda_n$. Moreover, if $r = \mathrm{ord}(\lambda)$, there is a decomposition*

$$T_{[\omega]}\mathbb{B}^n_\Lambda \cong \mathrm{Hom}\left(\mathbb{W}, \left(\bigoplus_i \mathbb{V}_r^{(k_i)}\right)/\mathbb{W}\right) \oplus \bigoplus_j \mathrm{Hom}\left(\mathbb{W}, \mathbb{V}_{d_j}^{(k_j)}\right),$$

where $\mathbb{V}_d^{(k)} = V_d^{(k)} \otimes_K \mathbb{C}$ *are the complexifications of the irreducible components of* Λ_K *as a* g*-module via Proposition 3.2.5.*

Proof. The statement about eigenvalues follows from the above discussion and linear algebra via the canonical isomorphism $T_{[\omega]}\mathbb{B}^n_\Lambda \cong \mathrm{Hom}(\mathbb{W}, \Lambda_{\mathbb{C}}/\mathbb{W})$.

The decomposition follows by applying eq. (4.2.1) and then taking $\bigoplus_{i=1}^{\mu} \mathbb{V}_r^{(k_i)}$ to be the complexification of all $V_r^{(k)}$ appearing in the decomposition. Then $\mathbb{W} \subset \bigoplus_{i=1}^{\mu} \mathbb{V}_r^{(k_i)}$, so $\mathbb{V}_{d_j}^{(k_j)}/\mathbb{W} = \mathbb{V}_{d_j}^{(k_j)}$. The stated decomposition then follows by $\mathrm{Hom}\left(\mathbb{W}, \bigoplus V_{d_j}^{(k_j)}\right) \cong \bigoplus \mathrm{Hom}\left(\mathbb{W}, V_{d_j}^{(k_j)}\right)$ since we are working with finite dimensional vector spaces. □

Remark 4.2.2. One could follow the strategy of [Beh11] or [GHS07] where they decompose Λ into two g-invariant sublattices S and T. This also provides a method for extracting the eigenvalues for the action. We choose to work without this decomposition since our strategy generalizes to the case of generalized ball quotients in Chapter 7.

4.3 Singularities Away from the Branch Divisor

We now apply the Reid–Tai criterion to V/G where $V = T_{[\omega]}\mathbb{C}H^n = \mathrm{Hom}(\mathbb{W}, \mathbb{C}^{n+1}/\mathbb{W})$. As before, fix $[\omega] \in \mathbb{B}^n_\Lambda$, $g \in \mathrm{Stab}_\Gamma([\omega])$, and let λ be the eigenvalue of g acting on $\omega \in \Lambda_{\mathbb{C}}$. We begin by studying points in $\mathbb{B}^n_\Lambda/\Gamma$ that are not contained in the branch divisor of the quotient map $\pi : \mathbb{B}^n_\Lambda \to \mathbb{B}^n_\Lambda/\Gamma$. The ramification divisors of π are the fixed loci of elements of Γ acting by quasi-reflections. Indeed, $[\omega] \in \mathbb{B}^n_\Lambda$ is ramified for π if the stabilizer of $[\omega]$ in Γ is nontrivial, i.e., $\mathrm{stab}_\Gamma([\omega]) \neq 1$. But an element $g \in \Gamma$ locally fixes a divisor pointwise if and only if g acts as a quasi-reflection. Therefore, $[\omega]$ is contained in a ramification divisor if and only if $[\omega]$ is fixed by a quasi-reflection. Therefore, we assume for now that no power of g acts by quasi-reflection.

Recall that $r = \mathrm{ord}(\lambda)$. Following the notation of Lemma 4.2.1, the eigenvalues of g acting on V arise from either a $V_r^{(k_i)}$ or a $V_{d_j}^{(k_j)}$. Our strategy is to consider what values of r and d_j may lead to $\sum(g) < 1$. We begin by studying the contribution of $V_r^{(k_i)}$ to $\sum(g)$.

Without loss of generality, let $\lambda = \zeta_r^{k_1}$ be the eigenvalue of g acting on $\mathbb{W}$, where $\zeta_r = e^{2\pi i/r}$.

Then the eigenvalue on the dual $\mathbb{W}^\vee$ is given by $\zeta_r^{k_2}$ where $k_2 = r - k_1$. We will be looking at the eigenvalues of the action of g on the subspace $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^{(k)}/\mathbb{W}) \subseteq T_{[\omega]}\mathbb{B}^n_\Lambda$ given in the decomposition of Lemma 4.2.1. By Proposition 3.2.5, the eigenvalues of g acting on $\mathbb{V}_r^{(k)}$ are given by either all rth roots of unity or the rth roots of unity $\zeta_r^{k_i}$ where $(k_i, r) = 1$ and $k_i \equiv k_1 \pmod 3$. We let

$$A = \{0 \leq k_i < r : (k_i, r) = 1, k_i \equiv k_1 \pmod 3\}$$

be the set of all exponents k_i giving the eigenvalues of g acting on $\mathbb{V}_r^{(k)}$. Therefore the eigenvalues of g acting on $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^{(k)}/\mathbb{W})$ are given by $\zeta_r^{k_2}\zeta_r^{k_i}$ where $k_i \in A - \{k_1\}$.

For $q \in \mathbb{Q}$, let $\{q\}$ denote the fractional part of q. To be precise, $\{q\}$ is the representative of the class $\overline{q} \in \mathbb{Q}/\mathbb{Z}$ such that $0 \leq \{q\} < 1$. Therefore, we have

$$\sum(g) \geq \sum_{k_i \in A - \{k_1\}} \left\{ \frac{k_2 + k_i}{r} \right\} \geq \sum_{j=1}^{\varphi(r)} \frac{j}{r}. \tag{4.3.1}$$

if $|A| = \varphi(r)$ and

$$\sum(g) \geq \sum_{k_i \in A - \{k_1\}} \left\{ \frac{k_2 + k_i}{r} \right\} \geq \sum_{j=1}^{\frac{\varphi(r)}{2} - 1} \frac{j}{r}. \tag{4.3.2}$$

if $|A| = \varphi(r)/2$.

Lemma 4.3.1. *Suppose that λ is the eigenvalue of $g \in \mathrm{Stab}_\Gamma([\omega])$ acting on $\mathbb{W}$ and $|\lambda| = r$. Then $\sum(g) \geq 1$ if $r \neq 1, 2, 3, 4, 6, 12$.*

Proof. Either $|A| = \varphi(r)$ or $|A| = \varphi(r)/2$, depending on whether $3 \nmid r$ or $3 \mid r$ by Proposition 3.2.5. We proceed first in the case where $3 \mid r$, so $|A| = \varphi(r)/2$. Therefore,

$$\sum(g) \geq \sum_{k_i \in A - \{k_1\}} \left\{ \frac{k_2 + k_i}{r} \right\} \geq \sum_{j=1}^{\frac{\varphi(r)}{2} - 1} \frac{j}{r}.$$

Let $r = 3^a q_1^{b_1} \cdots q_s^{b_s}$ be the prime factorization of r. Then

$$\frac{\varphi(r)}{2} = 3^{a-1} \prod_{i=1}^{s} \left(q_i^{b_i} - q_i^{b_i - 1} \right).$$

Therefore,

$$\sum_{j=1}^{\frac{\varphi(r)}{2} - 1} \frac{j}{r} = \frac{\left(3^{a-1} \prod_{i=1}^{s} \left(q_i^{b_i} - q_i^{b_i-1} \right) - 1\right) \left(3^{a-1} \prod_{i=1}^{s} \left(q_i^{b_i} - q_i^{b_i-1} \right)\right)}{2r}.$$

We will show that this quantity is ≥ 1 for most values of r. Equivalently, we want to show that the following inequality holds for most values of r:

$$\left(3^{a-1}\prod_{i=1}^{s}\left(q_i^{b_i}-q_i^{b_i-1}\right)-1\right)\left(3^{a-1}\prod_{i=1}^{s}\left(q_i^{b_i}-q_i^{b_i-1}\right)\right)\geq 2\cdot 3^a q_1^{b_1}\cdots q_s^{b_s}.$$

By rearranging terms, the above inequality is equivalent to the following inequality:

$$3^{a-2}\prod_{i=1}^{s}\left(q_i^{b_i}-2q_i^{b_i-1}+q_i^{b_i-2}\right)\geq 2+\frac{1}{3}\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right). \tag{4.3.3}$$

In summary, we want to show that 4.3.3 holds for most values of r. The following lemma is useful for analyzing this inequality.

Lemma 4.3.2. *Let q be prime and $b \geq 1$ be an integer. Then $q^b - 2q^{b-1} + q^{b-2} < 1$ if and only if $q = 2$ and $b = 1$. In this case, $q^b - 2q^{b-1} + q^{b-2} = \frac{1}{2}$. Also, $q^b - 2q^{b-1} + q^{b-2} = 1$ if and only if $q = 2$ and $b = 2$.*

Moreover, if $q \geq 5$, then $q^b - 2q^{b-1} + q^{b-2} > 4$.

Proof. All statements follow by $q^b - 2q^{b-1} + q^{b-2} = q^{b-2}(q-1)^2$. We include details to be complete.

If $q > 2$, then $q^{b-2}(q-1)^2 \geq 3^{b-2}2^2 \geq 1$ for all $b \geq 1$. If $q = 2$ and $b \geq 2$, then $2^{b-2}2^2 \geq 1$. If $q = 2$ and $b = 1$, then $q^{b-2}(q-1)^2 = 1/2 < 1$.

For the second statement in the lemma, note that $q^{b-2}(q-1)^2 = 1$ if and only if $(q-1)^2 = q^{2-b}$. If $q > 2$, then $(q-1)^2 > 1$. If $b > 2$, then $q^{2-b} < 1$. The case of $q = 2$ and $b = 1$ leads to $q^{b-2}(q-1)^2 = 1/2$. Therefore, $(q-1)^2 = q^{2-b}$ if and only if $q = 2$ and $b = 2$.

If $q \geq 5$, then $q^{b-2}(q-1)^2 \geq 5^{1-2}4^2 > 4$. $\square$

Suppose first that $a \geq 4$. Then,

$$3^{a-2}\prod_{i=1}^{s}\left(q_i^{b_i}-2q_i^{b_i-1}+q_i^{b_i-2}\right)\geq 3^2\prod_{i=1}^{s}\left(q_i^{b_i}-2q_i^{b_i-1}+q_i^{b_i-2}\right).$$

But,

$$3^2\prod_{i=1}^{s}\left(q_i^{b_i}-2q_i^{b_i-1}+q_i^{b_i-2}\right)\geq\frac{9}{2}\geq 2+\frac{1}{3}\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)$$

by Lemma 4.3.2. Therefore $\sum(g) \geq 1$ if $a \geq 4$.

Now suppose $a = 3$. By Lemma 4.3.2 and the fact that each prime q_i is distinct and not equal to 3, if $r \neq 54$, then

$$3\prod_{i=1}^{s}\left(q_i^{b_i} - 2q_i^{b_i-1} + q_i^{b_i-2}\right) \geq 3 \geq 2 + \frac{1}{3}\prod_{i=1}^{s}\left(1 - \frac{1}{q_i}\right).$$

Now suppose $a = 2$. By Lemma 4.3.2 and the fact that each prime q_i is distinct and not equal to 3, if $r \neq 9, 18, 36, 72, 90$, then

$$\prod_{i=1}^{s}\left(q_i^{b_i} - 2q_i^{b_i-1} + q_i^{b_i-2}\right) \geq 4 \geq 2 + \frac{1}{3}\prod_{i=1}^{s}\left(1 - \frac{1}{q_i}\right).$$

Now suppose $a = 1$. By computer calculation, if $r \neq 3, 6, 12, 15, 21, 24, 30, 42, 48, 60, 66,$ 78,84,120, then

$$\frac{1}{3}\prod_{i=1}^{s}\left(q_i^{b_i} - 2q_i^{b_i-1} + q_i^{b_i-2}\right) \geq 3 \geq 2 + \frac{1}{3}\prod_{i=1}^{s}\left(1 - \frac{1}{q_i}\right).$$

For the cases of $r = 3, 6, 9,\ 12, 15, 18,\ 21, 24, 30,\ 36, 42,\ 48, 54, 60,\ 66, 72, 78,\ 84, 90, 120,$ we use computer calculation to compute all possible values of

$$\sum_{k_i \in A - \{k_1\}} \left\{\frac{k_2 + k_i}{r}\right\}$$

corresponding to different choices of eigenvalue λ. We find that $\sum(g) \geq 1$ in all cases except $r \in \{3, 6, 12\}$.

Now we consider the case where $3 \nmid r$. Therefore,

$$\begin{aligned}
\sum(g) &\geq \sum_{k_i \in A - \{k_1\}} \left\{\frac{k_2 + k_i}{r}\right\} \\
&\geq \sum_{j=1}^{\varphi(r)-1} \frac{j}{r} \\
&= \frac{(\varphi(r) - 1)\varphi(r)}{2r} \\
&= \frac{\prod_{i=1}^{s}\left(q_i^{b_i} - q_i^{b_i-1}\right)^2 - \prod_{i=1}^{s}\left(q_i^{b_i} - q_i^{b_i-1}\right)}{2r}.
\end{aligned}$$

By the same manipulation as before we have that this expression is ≥ 1 if and only if

$$\prod_{i=1}^{s}\left(q_i^{b_i} - 2q_i^{b_i-1} + q_i^{b_i-2}\right) \geq 2 + \prod_{i=1}^{s}\left(1 - \frac{1}{q_i}\right).$$

By Lemma 4.3.2, we have that if $r \neq 1, 2, 4, 8, 10$ then

$$\prod_{i=1}^{s} \left(q_i^{b_i} - 2q_i^{b_i - 1} + q_i^{b_i - 2} \right) \geq 3 \geq 2 + \prod_{i=1}^{s} \left(1 - \frac{1}{q_i} \right).$$

We use computer calculation to verify that in the cases of $r = 8, 10$ that the minimum contribution given by

$$\sum_{k_i \in A - \{k_1\}} \left\{ \frac{k_2 + k_i}{r} \right\}$$

is ≥ 1. $\square$

Example 4.3.3. If $r = 7$ then $V_r^{(k)} = V_7^{(0)}$ with eigenvalues given by the primitive seventh roots of unity. One of these eigenvalues is λ and the others we will call $\lambda_1, \ldots, \lambda_5$. On $T_{[\omega]}\mathbb{B}^n_\Lambda$, the eigenvalues contributing to $\sum(g)$ will then be given by $\lambda^{-1}\lambda_1, \ldots, \lambda^{-1}\lambda_5$. One can then verify that regardless of the choice of λ, there will always be two eigenvalues among $\lambda^{-1}\lambda_1, \ldots, \lambda^{-1}\lambda_5$ that are nontrivial and multiplicative inverses of one another. This guarantees that $\sum(g) \geq 1$.

Now that we understand the contributions of $V_r^{(k)}$ to $\sum(g)$, we consider the contributions of $V_{d_j}^{(k_j)}$ to $\sum(g)$.

Lemma 4.3.4. *Let $g \in G$ not act by quasi-reflection and*

$$\Lambda_{\mathbb{Q}(\zeta_3)} = \bigoplus_{i=1}^{\mu} V_r^{(k_i)} \oplus \bigoplus_{j=1}^{s} V_{d_j}^{(\ell_j)}$$

be the decomposition of Λ given in Lemma 4.2.1.

If $r \notin \{1, 2, 3, 4, 6, 12\}$ or $d_j \notin \{1, 2, 3, 4, 6, 12\}$ for some j, then $\sum(g) \geq 1$.

Proof. We adopt the notation of the proof of [Beh11, Theorem 2]. Let m be the order of g and ζ be a primitive mth root of unity. On $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_d) \subset V$ the element g has eigenvalues $\zeta^{\frac{mc}{r}}\zeta^{\frac{mk_i}{d}}$ for a fixed $0 < c < r$ with $(c, r) = 1$, and k_i coming from the eigenvalues of $\mathbb{V}_d$. So we have $(k_i, d) = 1$ and, if $3 \mid d$, $k_i \equiv \alpha \pmod 3$ where $\alpha \in \{1, 2\}$, by Proposition 3.2.5. We will use A to denote the set of k_i. Therefore the contribution of g on $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_d)$ is given by

$$\sum_{k_i \in A} \left\{ \frac{c}{r} + \frac{k_i}{d} \right\}.$$

This sum is greater than or equal to $\displaystyle\sum_{j=1}^{\frac{\varphi(d)}{2}} \frac{j}{d}$ if $d \notin \varphi^{-1}(\{2,4,6,8\})$ and may contribute less than 1 if $d = 1, 2, \ldots, 10, 12, 14, 15, 16, 18, 20, 22, 24, 26, 28, 30, 36, 40, 42, 48, 54, 60, 66, 84, 90$ [Beh11, (6)].

By Lemma 4.3.1, we may assume that $r \in \{1, 2, 3, 4, 6, 12\}$. So we can compute a lower bound for the contribution of $\mathbb{V}_d$ to $\sum(g)$ as

$$\min(d) = \min_{r\in\{1,2,3,4,6,12\}} \min_{\substack{0<c<r\\(c,r)=1}} \sum_{\substack{(k_i,d)=1\\0<k_i<d}} \left\{\frac{c}{r} + \frac{k_i}{d}\right\}$$

if $3 \nmid d$ and

$$\min(d) = \min_{r\in\{1,2,3,4,6,12\}} \min_{\substack{0<c<r\\(c,r)=1}} \min_{\alpha\in\{1,2\}} \sum_{\substack{(k_i,d)=1\\0<k_i<d\\k_i\equiv\alpha \mod 3}} \left\{\frac{c}{r} + \frac{k_i}{d}\right\}$$

if $3 \mid d$.

Computer calculation yields that $\min(d) < 1$ only if $d \in \{1, 2, 3, 4, 6, 12\}$. $\square$

Let μ denote the number of $V_r^{(k)}$ appearing in the decomposition of Λ_K. Since we are searching for elements with $\sum(g) < 1$, we may assume that the other $V_d^{(j)}$ appearing in the decomposition of Λ_K must have $d \in \{1, 2, 3, 4, 6, 12\}$. Therefore we have the following dimension count

$$\dim \mathbb{V}_r \cdot \mu + \nu_1 + \nu_2 + \nu_3 + 2\nu_4 + \nu_6 + 2\nu_{12} = n + 1, \tag{4.3.4}$$

where ν_d denotes the multiplicity of $\mathbb{V}_d$ in Λ_K. Recall that the dimensions of $V_3^{(j)}, V_6^{(j)}, V_{12}^{(j)}$ are $1, 1, 2$ respectively because 3 divides 3, 6, and 12.

Lemma 4.3.5. *If $r \in \{4, 12\}$ and $n \geq 7$, then $\sum(g) \geq 1$.*

Proof. We proceed by cases on the value of r.

If $r = 4$, then we get that

$$\nu_1 + \nu_2 + \nu_3 + 2\nu_4 + \nu_6 + 2\nu_{12} = n - 1.$$

The minimal contributions to the RT sum for each d are

d	contribution
1	1/4
2	1/4
3	1/12
4	1/2
6	1/12
12	5/6

Table 4.1: Minimal contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 4$

We compute that on $\mathbb{V}_{(4)}^{(0)}$ we get a minimal contribution of 1/2. Therefore we can assume there are no components V_d with $d = 4, 12$ so the dimension count becomes

$$\nu_1 + \nu_2 + \nu_3 + \nu_6 = n - 1.$$

Each summand contributes at a minimum 1/12 to $\sum(g)$, so if $n \geq 7$ then $\sum(g) \geq 1$.

If $r = 12$, then the minimal contributions are

d	contribution
1	1/12
2	1/12
3	1/12
4	5/6
6	1/12
12	1/2

Table 4.2: Minimal contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 12$

and the minimal contribution on $\mathbb{V}_{12}^{(k)}$ is 1/2. Again, we can assume there are no components with $d = 4, 12$. So we obtain the dimension formula

$$\nu_1 + \nu_2 + \nu_3 + \nu_6 = n - 1.$$

Note that the dimension of $\mathbb{V}_{12}$ is $\phi(12)/2 = 2$ since $3 \mid 12$. Therefore if $n \geq 7$, $\sum(g) \geq 1$. $\square$

The strategy of the above proof fails to obtain a dimension bound forcing singularities to be canonical in the cases of $r = 1, 2, 3, 6$. The following example highlights the obstruction.

Example 4.3.6. Consider $G = \mathbb{Z}/3\mathbb{Z}$ acting on $\Lambda \cong \mathcal{O}^{n+1}$ by

$$1 \cdot (z_0, z_1, z_2, \ldots, z_n) = (\zeta_3^{-1} z_0, \zeta_3^{-1} z_1, \zeta_3 z_2, \ldots, \zeta_3 z_n).$$

The point $[\omega] = [1, \ldots, 0, 0] \in \mathbb{B}^n_\Lambda$ is fixed by this action. We will study $\sum(g)$ where $g = 1$.

Clearly the action on $\mathbb{C}^{n+1}$ has eigenvalues $\zeta_3^{-1}, \zeta_3^{-1}, \zeta_3, \ldots, \zeta_3$. One of the ζ_3 eigenvalues is lost when we quotient by $\mathbb{W}$. We note that $1 \cdot \omega = \zeta_3 \omega$. Therefore the eigenvalues of the action on $T_{[\omega]}\mathbb{B}^n_\Lambda$ are given by multiplying all of the previous eigenvalues by ζ_3^{-1}. This yields $\zeta_3, \zeta_3, 1, 1, \ldots, 1$. Note that g is not acting as a quasi-reflection, $\sum(g) = \frac{2}{3} < 1$, and the dimension n can be arbitrarily large.

This singularity is locally isomorphic to the affine cone over the rational normal curve of degree 3. Indeed, this is coming from the quotient of $\mathbb{C}^2$ by the action $(x, y) \mapsto (\zeta_3 x, \zeta_3 y)$, whose ring of functions is given by $\mathbb{C}[x^3, x^2y, xy^2, y^3]$. This singularity is known to be noncanonical.[1]

Definition 4.3.7. A finite cyclic quotient singularity $p \in X$ is of type $\frac{1}{n}(a_1, \ldots, a_k)$ if it is locally analytically isomorphic to $(\mathbb{C}^k/\langle g \rangle) \times \mathbb{C}^{m-k}$ where g acts on $\mathbb{C}^k$ as the diagonal matrix with entries $(\zeta_n^{a_1}, \ldots, \zeta_n^{a_k})$ where $\zeta_n = e^{2\pi i/n}$. We call $\mathbb{C}^{m-k}$ the smooth component at p.

Remark 4.3.8. Finite cyclic quotient singularities can also be defined via toric geometry. A singularity of type $\frac{1}{n}(a_1, \ldots, a_k)$ is obtained as a toric singularity by considering the standard cone and lattice for $\mathbb{A}^m$ and then considering the quotient toric variety obtained by taking the same cone but instead taking the lattice generated by the standard lattice and the point $\left(\frac{a_1}{n}, \ldots, \frac{a_k}{n}\right)$.

[1] By a small change to this example, one can see that [Beh11, Lemma 7] does not hold in the case $D = -3$. Indeed, the action $g \cdot (z_0, z_1, z_2, \ldots, z_n) = (z_0, \zeta_3 z_1, \zeta_3 z_2, z_3, \ldots, z_n)$ would have $r = 1$ and be defined over $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$ with $\sum(g) < 1$.

Example 4.3.9. The affine quadric cone $xy - z^2 = 0$ has a singularity of type $\frac{1}{2}(1,1)$.

Example 4.3.10. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{C}^n$ by

$$(z_1, z_2, z_3, \ldots, z_n) \mapsto (-z_1, -z_2, z_3, \ldots, z_n)$$

where $n \geq 2$. The quotient $\mathbb{C}^n/G$ has a singularity of type $\frac{1}{2}(1,1)$ at the origin with smooth components isomorphic to $\mathbb{C}^{n-2}$.

Example 4.3.11. Consider the action of $G = \mathbb{Z}/3\mathbb{Z}$ on Λ via

$$1 \cdot (z_0, z_1, z_2, \ldots, z_n) = (\zeta_3^{-1} z_0, \zeta_3^{-1} z_1, \zeta_3 z_2, \ldots, \zeta_3 z_n)$$

as in Example 4.3.6. Then $\mathbb{B}^n_\Lambda/G$ has a singularity of type $\frac{1}{3}(1,1)$ at $[1 : 0 : \cdots : 0]$.

Proposition 4.3.12. *If $\mathbb{B}^n_\Lambda$ is defined by the Eisenstein integers, $n \geq 7$, $G = \mathrm{stab}_\Gamma([\omega])$ does not act with quasi-reflections, and $\overline{[\omega]} \in \mathbb{B}^n_\Lambda/\Gamma$ is singular, then the singularity at $\overline{[\omega]}$ is either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g \rangle \subseteq G$ so that $\mathbb{C}^n/\langle g \rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$.*

Proof. By Lemma 4.3.4 and Lemma 4.3.5 we reduce to the cases where $r \in \{1,2,3,6\}$ and $d \in \{1,2,3,4,6,12\}$. We proceed by cases on r.

Assuming that $r = 1$, the contribution of each V_r toward $\sum(g)$ is 0. The contributions of each $V_d^{(\ell)}$ is given by the following table where one entry indicates that we only have $V_d^{(0)}$ and two entries indicates the contributions of $V_d^{(1)}$ and $V_d^{(2)}$, respectively.

d	contribution
1	0
2	1/2
3	1/3, 2/3
4	1
6	1/6, 5/6
12	2/3, 4/3

Table 4.3: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 1$

From here we obtain a list of 14 possible cases for $\bigoplus V_{d_j}^{(k_j)}$ where $\sum(g) < 1$.

(1) $V_2^{(0)} \oplus V_3^{(1)}$

(2) $V_2^{(0)} \oplus V_6^{(1)}$

(3) $V_2^{(0)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(4) $V_3^{(1)} \oplus V_3^{(1)}$

(5) $V_3^{(1)} \oplus V_3^{(1)} \oplus V_6^{(1)}$

(6) $V_3^{(1)} \oplus V_6^{(1)}$

(7) $V_3^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(8) $V_3^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(9) $V_3^{(2)} \oplus V_6^{(1)}$

(10) $V_6^{(1)} \oplus V_6^{(1)}$

(11) $V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(12) $V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(13) $V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$

(14) $V_6^{(1)} \oplus V_{12}^{(1)}$

In all of these cases we can include any number of $V_1^{(0)}$'s in order to achieve some arbitrary dimension. For instance, we can take $V_3^{(1)} \oplus V_3^{(1)} \oplus V_1^{(0)} \oplus \cdots V_1^{(0)}$ to obtain a singularity of type $\frac{1}{3}(1,1)$ of arbitrary dimension.

In cases (1), (2), (5), (6), and (9), g^k is a quasi-reflection for some k. After removing these cases, we see that (3) corresponds to type $\frac{1}{6}(1,1,3)$, (4) corresponds to type $\frac{1}{3}(1,1)$, (7) corresponds to type $\frac{1}{6}(1,1,2)$, (8) corresponds to type $\frac{1}{6}(1,1,1,2)$, (10) corresponds to type $\frac{1}{6}(1,1)$, (11) corresponds to type $\frac{1}{6}(1,1,1)$, (12) corresponds to type $\frac{1}{6}(1,1,1,1)$, (13) corresponds to type $\frac{1}{6}(1,1,1,1,1)$, and (14) corresponds to type $\frac{1}{12}(1,2,7)$.

For the remaining cases of $r = 2,3,6$, we give the table of contributions of $V_d^{(k)}$. It is then easy to verify that no new singularities arise using the same methods as for $r = 1$. For the cases of $r = 3$ and $r = 6$ we have included two tables corresponding to the two possible eigenvalues λ arising from either $V_r^{(1)}$ or $V_r^{(2)}$.

d	contribution
1	1/2
2	0
3	5/6, 1/6
4	1
6	2/3, 1/3
12	2/3, 4/3

Table 4.4: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 2$

d	contribution
1	$2/3$
2	$1/6$
3	$0, \ 1/3$
4	$4/3$
6	$5/6, \ 1/2$
12	$1, \ 2/3$

Table 4.5: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 3$ and $\lambda = \zeta_3$

d	contribution
1	$1/3$
2	$5/6$
3	$2/3, \ 0$
4	$2/3$
6	$1/2, \ 1/6$
12	$4/3, \ 1$

Table 4.6: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 3$ and $\lambda = \zeta_3^2$

d	contribution
1	5/6
2	1/3
3	1/6, 1/2
4	2/3
6	0, 2/3
12	4/3, 1

Table 4.7: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 6$ and $\lambda = \zeta_6$

d	contribution
1	1/6
2	2/3
3	1/2, 5/6
4	4/3
6	1/3, 0
12	1, 2/3

Table 4.8: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 6$ and $\lambda = \zeta_6^5$

□

Corollary 4.3.13. *In the setting of Proposition 4.3.12, if $\sum(g) < 1$ there exists a 1-dimensional $V_d^{(k)}$ contributing a nontrivial eigenvalue to the action of g on $T_{[\omega]}\mathbb{B}^n_\Lambda$.*

Proof. Consider all cases in the proof of Proposition 4.3.12 leading to $\sum(g) < 1$. □

When $n \leq 6$, we can also classify the possible noncanonical singularities arising from elements g with $r = 4$ or $r = 12$.

Proposition 4.3.14. *If $\mathbb{B}^n_\Lambda$ is defined by the Eisenstein integers, $G = \mathrm{stab}_\Gamma([\omega])$ does not act with quasi-reflections, and $\overline{[\omega]} \in \mathbb{B}^n_\Lambda/\Gamma$ is singular, then the singularity at $\overline{\omega}$ is either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g \rangle \subseteq G$ so that $\mathbb{C}^n/\langle g \rangle$ is of type $\frac{1}{3}(1,1), \frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1), \frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3), \frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$, or of type $\frac{1}{12}(1,1,3,6)$, $\frac{1}{12}(1,1,6)$, $\frac{1}{12}(1,1,1,6)$, $\frac{1}{12}(1,1,1,1,6)$, or $\frac{1}{12}(1,1,1,1,1,6)$. Moreover, if the singularity at $\overline{[\omega]}$ is from the first list, then the eigenvalue of g acting on $\omega \in \mathbb{C}^{n+1}$ has order dividing 6. If the singularity is on the second list, then the eigenvalue of g acting on ω has order 4 or 12.*

Remark 4.3.15. Note that in both the case of $r = 4$ and $r = 12$, we do not have a representation with trivial contribution to the Reid–Tai sum. Therefore the dimension of the singularity must match the dimension of the ball. For example, if a singularity of type $\frac{1}{12}(1,1,3,6)$ appears, then we must be considering a ball quotient defined on $\mathbb{B}^4_\Lambda$.

Proof. If $r = 4$, then the following tables describe the contributions of each V_d.

d	contribution
1	3/4
2	1/4
3	1/12, 5/12
4	1/2
6	11/12, 7/12
12	7/6, 5/6

Table 4.9: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 4$ and $\lambda = i$

d	contribution
1	1/4
2	3/4
3	7/12, 11/12
4	1/2
6	5/12, 1/12
12	7/6, 5/6

Table 4.10: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 4$ and $\lambda = -i$

The contribution of V_r in either case is 1/2. We only consider the case where $\lambda = i$. The reader can verify that the same singularities arise when $\lambda = -i$. We get the following list of possible representations contributing $\sum(g) < 1$.

(1) $V_4^{(0)} \oplus V_2^{(0)}$

(2) $V_4^{(0)} \oplus V_2^{(0)} \oplus V_3^{(1)}$

(3) $V_4^{(0)} \oplus V_2^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)}$

(4) $V_4^{(0)} \oplus V_3^{(1)}$

(5) $V_4^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)}$

(6) $V_4^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$

(7) $V_4^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$

(8) $V_4^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$

(9) $V_4^{(0)} \oplus V_3^{(2)}$

Now we claim that (1), (2), (4), and (9) all have that g^k is a quasi-reflection. For instance, if g gives (1) as a representation, then g^2 will act as a quasi-reflection (the eigenvalues of g are -1 and i on $T_{[\omega]}\mathbb{B}^n_\Lambda$ from V_4^ω and V_2 respectively, so g^2 will be a quasi-reflection).

The other representations yield cyclic quotient singularities of type $\frac{1}{12}(1,1,3,6)$, $\frac{1}{12}(1,1,6)$, $\frac{1}{12}(1,1,1,6)$, $\frac{1}{12}(1,1,1,1,6)$, or $\frac{1}{12}(1,1,1,1,1,6)$.

If $r = 12$, the contribution of V_r to the Reid–Tai sum is $1/2$ regardless of the choice of k or Λ. Although there are four cases to verify, we only show the table of values for the case $V_r = V_{12}^{(2)}$ and $\lambda = \zeta_{12}^{11}$.

d	contribution
1	1/12
2	7/12
3	5/12, 3/4
4	7/6
6	1/4, 11/12
12	5/6, 1/2

Table 4.11: Contributions of $V_d^{(k_i)}$ to $\sum(g)$ when $r = 12$ and $\lambda = \zeta_{12}^{11}$

From here we obtain a similar list to the case of $r = 4$:

(1) $V_{12}^{\omega} \oplus V_6^{(1)}$

(2) $V_{12}^{\omega} \oplus V_6^{(1)} \oplus V_1$

(3) $V_{12}^{\omega} \oplus V_6^{(1)} \oplus V_1 \oplus V_1$

(4) $V_{12}^{\omega} \oplus V_1$

(5) $V_{12}^{\omega} \oplus V_1 \oplus V_1$

(6) $V_{12}^{\omega} \oplus V_1 \oplus V_1 \oplus V_1$

(7) $V_{12}^{\omega} \oplus V_1 \oplus V_1 \oplus V_1 \oplus V_1$

(8) $V_{12}^{\omega} \oplus V_1 \oplus V_1 \oplus V_1 \oplus V_1 \oplus V_1$

(9) $V_{12}^{\omega} \oplus V_3^{(1)}$

By the same computation, (1), (2), (4), and (9) all have that g^k is a quasi-reflection. The other representations yield cyclic quotient singularities of type $\frac{1}{12}(1,1,3,6)$, $\frac{1}{12}(1,1,6)$, $\frac{1}{12}(1,1,1,6)$, $\frac{1}{12}(1,1,1,1,6)$, or $\frac{1}{12}(1,1,1,1,1,6)$. □

4.4 Singularities on the Branch Divisor

Following [Beh11, Section 3] and [GHS07, Section 2], we now consider the case where $\langle g \rangle$ acts with quasi-reflections. By the discussion at the beginning of Section 4.3, we are now considering singularities contained in the branch divisor of $\mathbb{B}^n_\Lambda/\Gamma$.

Let $V = T_{[\omega]}\mathbb{B}^n_\Lambda$. We still want to determine if $V/\langle g \rangle$ is canonical and consider the eigenvalues leading to noncanonical singularities. In order to accomplish this, we let $H \subseteq \langle g \rangle$ be the subgroup generated by quasi-reflections. By [Che55, (A)], V/H is nonsingular. By [Ish18, Theorem 7.4.8], $\langle g \rangle/H$ acts on V/H without quasi-reflections. Moreover,

$$V/H \cong (V/H)/(\langle g \rangle/H).$$

Therefore, it suffices to consider the eigenvalues of $\langle g \rangle/H$ acting on V/H and apply the Reid–Tai criterion in this setting.

Since $\langle g \rangle$ acts with quasi-reflections, there exists k minimal such that g^k is a quasi-reflection. It is not true in general that $H = \langle g^k \rangle$.

Example 4.4.1. Consider $\langle g \rangle = \mathbb{Z}/6\mathbb{Z}$ acting on Spec $\mathbb{C}[x,y]$ as g acting by $(x,y) \mapsto (\zeta_3 x, -y)$. Then g^2 acts as the quasi-reflection $(x,y) \mapsto (\zeta_3^2 x, y)$. But g^3 also acts as a quasi-reflection $(x,y) \mapsto (x,-y)$. Therefore, $H = \langle g^2, g^3 \rangle = \langle g \rangle \neq \langle g^2 \rangle$.

We will proceed by restricting the order of the group H by studying the quasi-reflections that can appear on arithmetic ball quotients defined by the Eisenstein integers. We first turn our attention toward understanding how Λ_K decomposes as a g-module when we assume $g^k = h$ is a quasi-reflection with k minimal. We start by stating two results of Behrens that apply directly to the Eisenstein case.

Lemma 4.4.2 ([Beh11, Proposition 2, case (iii)])**.** *Let* $h = g^k$ *be a quasi-reflection on* $T_{[\omega]}\mathbb{B}^n_\Lambda$ *for* $g \in \mathrm{Stab}_\Gamma([\omega])$*,* $n \geq 2$*, and* Λ *a lattice defined over the Eisenstein integers. As a* g*-module we have a decomposition of the form*

$$\Lambda_K \cong V_{m_0}^{(b_0)} \oplus \bigoplus_j V_{m_j}^{(b_j)}$$

for some $m_i \in \mathbb{N}$*. Then for all* $j \geq 1$ *either*

(1) $(m_0, k) = m_0$ *and* $\ell \cdot (m_j, k) = m_j$ *with* $\ell \in \{2, 3, 6\}$*, or*

(2) $\ell \cdot (m_0, k) = m_0$ *and* $(m_j, k) = m_j$ *with* $\ell \in \{2, 3, 6\}$*.*

Lemma 4.4.3 ([Beh11, Corollary 3])**.** *If* Λ *is a lattice over the Eisenstein integers, then the quasi-reflections on* $T_{[\omega]}\mathbb{B}^n_\Lambda$ *are induced by elements* $h \in \mathrm{U}(\Lambda)$ *such that* $h^6 \sim I$*; i.e.,* h^6 *acts as the identity on* $T_{[\omega]}\mathbb{B}^n_\Lambda$*.*

Example 4.4.4. This example shows that not every decomposition satisfying criteria (1) or (2) of Lemma 4.4.2 arises from some element g such that g^k acts as a quasi-reflection. We will assume that g^k acts as a quasi-reflection, construct a possible g-module decomposition satisfying the numerical conclusion of Lemma 4.4.2, then show that this g-module decomposition would force g^k to not act as a quasi-reflection.

Assume Λ has rank 12 and $g \in \mathrm{U}(\Lambda)$ acts such that g^7 acts on $T_{[\omega]}\mathbb{B}^{11}_\Lambda$ as a quasi-reflection. We are therefore assuming $k = 7$ in this example. Let $\ell = 3$, $m_0 = 7$, and $m_j = 21$. These integers, along with $k = 7$, satisfy condition (1) of Lemma 4.4.2. Let's assume that Λ_K decomposes as a g-module as

$$\Lambda_K \cong V_7^{(0)} \oplus V_{21}^{(1)}.$$

The eigenvalues of g acting on Λ_K are then $\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6, \zeta_{21}, \zeta_{21}^4, \zeta_{21}^{10}\zeta_{21}^{13}, \zeta_{21}^{16}, \zeta_{21}^{19}$, so the eigenvalues of $h = g^7$ acting on Λ_K are $1, 1, 1, 1, 1, 1, \zeta_3, \zeta_3, \zeta_3, \zeta_3, \zeta_3, \zeta_3$. It is clearly impossible for such an element to act on $T_{[\omega]}\mathbb{B}^n_\Lambda$ as a quasi-reflection. Therefore, Λ_K cannot decompose as a g-module as given above and have g^7 act as a quasi-reflection, even though the given g-module decomposition satisfies the numerical conclusion of condition (1).

The previous example indicates that we may be able to sharpen Lemma 4.4.2. Note that if h is a quasi-reflection on $T_{[\omega]}\mathbb{B}^n_\Lambda$, then the eigenvalues of h on Λ_K are all equal except for one distinguished eigenvalue.

Proposition 4.4.5. *Let $h = g^k$ be a quasi-reflection on $T_{[\omega]}\mathbb{B}^n_\Lambda$ for $g \in \mathrm{Stab}_\Gamma([\omega])$, $n \geq 2$, and Λ a lattice defined over the Eisenstein integers. As a g-module, Λ_K decomposes as*

$$\Lambda_K \cong V_m^{(b)} \oplus \bigoplus_j V_{m_j}^{(b_j)}$$

with $m \in \{1,2,3,6\}$ and $\dim \bigoplus_j V_{m_j}^{(b_j)} > 1$.

Moreover, the distinguished eigenvalue of h comes from $V_m^{(b)}$.

Proof. We proceed in the two cases of Lemma 4.4.2.

First assume we are in case (1), so that

$$\Lambda_K \cong V_{m_0}^{(b_0)} \oplus \bigoplus_j V_{m_j}^{(b_j)}$$

with $(m_0, k) = m_0$ and $\ell(m_j, k) = m_j$ with $\ell \in \{2,3,6\}$.

We will show that $m_0 \notin \{1,2,3,6\} \Rightarrow \dim \bigoplus_j V_{m_j}^{(b_j)} = 1$. Note that if $m_0 \in \{1,2,3,6\}$, then we can take $m = m_0$ to satisfy the conclusion. If $\dim \bigoplus_j V_{m_j}^{(b_j)} = 1$, then $\bigoplus_j V_{m_j}^{(b_j)} = V_{m_1}^{(b_1)}$ is 1-dimensional and so $m_1 \in \{1,2,3,6\}$ by Proposition 3.2.5.

Suppose that $m_0 \notin \{1,2,3,6\}$. So $\dim V_{m_0}^{(b_0)} > 1$. Since $(m_0, k) = m_0$, we have that $h = g^k$ acts trivially on $V_{m_0}^{(b_0)}$ (m_0 divides k and the eigenvalues of g on $V_{m_0}^{(b_0)}$ are primitive m_0th roots of unity, there the eigenvalues of g^k on $V_{m_0}^{(b_0)}$ are all 1). Since h acts as a quasi-reflection on the tangent space, all eigenvalues of h on Λ_K must be equal except for one distinguished eigenvalue λ corresponding to the nontrivial eigenvalue on the tangent space. Since $\dim V_{m_0}^{(b_0)} > 1$ and h acts trivially on $V_{m_0}^{(b_0)}$, we must have that all but one of the eigenvalues of h on $\bigoplus_j V_{m_j}^{(b_j)}$ are equal to 1.

Suppose for a contradiction that $\dim \bigoplus_j V_{m_j}^{(b_j)} > 1$. Then there exists some $V_{m_j}^{(b_j)}$ on which $h = g^k$ acts with a trivial eigenvalue. So g has an eigenvalue on $V_{m_j}^{(b_j)}$ that is a kth root of unity. By Proposition 3.2.5, the eigenvalues of g acting on $V_{m_j}^{(b_j)}$ are primitive m_jth roots of unity. Therefore

$(m_j, k) = m_j$, since $\zeta_{m_j}^k = 1$. But this is a contradiction since, by Lemma 4.4.2, $\ell(m_j, k) = m_j$ and $\ell \neq 1$. Therefore, $m_0 \notin \{1,2,3,6\} \Rightarrow \dim \bigoplus_j V_{m_j}^{(b_j)} = 1$.

If $m_0 \notin \{1,2,3,6\}$, then $\bigoplus_j V_{m_j}^{(b_j)} = V_m^{(b)}$ with $m \in \{1,2,3,6\}$ since these are the only possible one dimensional irreducible, faithful representations. Since h acts trivially on $V_{m_0}^{(b_0)}$ and $\dim V_{m_0}^{(b_0)} > 1$, we have that the distinguished eigenvalue of h comes from $V_m^{(b)}$.

If $m_0 \in \{1,2,3,6\}$, then we define $V_m^{(b)} = V_{m_0}^{(b)}$. Indeed, since $n \geq 2$, we have that $\dim \bigoplus_j V_{m_j}^{(b_j)} > 1$, so h cannot act with a trivial eigenvalue on any component of $\bigoplus_j V_{m_j}^{(b_j)}$ (otherwise, we obtain the same contradiction as before, that $(m_j, k) = m_j$). Thus, the eigenvalue of 1 by h on $V_{m_0}^{(b)}$ must be the distinguished eigenvalue.

So we have shown that the conclusion of the proposition holds for case (1) of Lemma 4.4.2.

Now assume we are in case (2) of Lemma 4.4.2, so that

$$\Lambda_K \cong V_{m_0}^{(b_0)} \oplus \bigoplus_j V_{m_j}^{(b_j)}$$

with $\ell(m_0, k) = m_0$ and $(m_j, k) = m_j$ with $\ell \in \{2,3,6\}$. We will show again that $m_0 \notin \{1,2,3,6\} \Rightarrow \dim \bigoplus_j V_{m_j}^{(b_j)} = 1$.

Suppose that $m_0 \notin \{1,2,3,6\}$. Since $\ell(m_0, k) = m_0$, the eigenvalues of g^k on $V_{m_0}^{(b_0)}$ are not trivial. Indeed, suppose for a contradiction that g^k acts trivially on $V_{m_0}^{(b_0)}$. The eigenvalues of g on $V_{m_0}^{(b_0)}$ are primitive m_0th roots of unity. The eigenvalues of g^k are then given by $\zeta_{m_0}^k$. We have that $\zeta_{m_0}^k = 1 \Rightarrow m_0 \mid k$, but $\ell(m_0, k) = m_0$ with $\ell \neq 1$. One of these nontrivial eigenvalues of h on $V_{m_0}^{(b_0)}$, we will denote it as μ, must match all other eigenvalues of h on Λ_K.

Suppose for a contradiction that $\dim \bigoplus_j V_{m_j}^{(b_j)} > 1$. Since $(m_j, k) = m_j$, we have that $h = g^k$ acts trivially on $\bigoplus_j V_{m_j}^{(b_j)}$. But this is a contradiction since all but one of the eigenvalues of h must be equal to μ, which is not equal to 1. Therefore, $m_0 \notin \{1,2,3,6\} \Rightarrow \dim \bigoplus_j V_{m_j}^{(b_j)} = 1$.

If $m_0 \in \{1,2,3,6\}$, we let $V_m^{(b)} = V_{m_0}^{(b_0)}$. Since $n \geq 2$, we have that $\dim \bigoplus_j V_{m_j}^{(b_j)} > 1$ and by the previous paragraph h acts trivially on $\bigoplus_j V_{m_j}^{(b_j)}$, therefore the distinguished eigenvalue of h comes from $V_m^{(b)}$.

If $m_0 \notin \{1,2,3,6\}$, then $\dim \bigoplus_j V_{m_j}^{(b_j)} = 1$, so $\bigoplus_j V_{m_j}^{(b_j)} = V_m^{(b)}$ with $m \in \{1,2,3,6\}$. As in the argument above, the eigenvalues of h acting on $V_{m_0}^{(b_0)}$ are nontrivial and the eigenvalue of h acting on $V_m^{(b)}$ is trivial. Therefore the distinguished eigenvalue of h must come from $V_m^{(b)}$. $\square$

By Lemma 4.4.3, we have that $|H| \mid 6$, where H is the subgroup of $\langle g\rangle$ generated by quasi-reflections. Indeed, $H = \langle t\rangle$, but $t = r_1^{a_1} \cdots r_j^{a_j}$ where $r_1, \dots, r_j$ are quasi-reflections. Therefore $t^6 = 1$.

Therefore, if $\langle g\rangle$ contains quasi-reflections we have the following two cases; either $H = \langle h\rangle$ is generated by the quasi-reflection g^k with k minimal, or $H = \langle h, h'\rangle$ is generated by quasi-reflections h and h' of order 3 and 2.

If we are in the first case, we proceed by applying the strategy found in [GHS07] for studying the eigenvalues of $\langle g\rangle/\langle h\rangle$ acting on $V/\langle h\rangle$. Let $\langle g\rangle$ act linearly on a $\mathbb{C}$-vector space V. If g is a quasi-reflection, then, considering V as $\mathbb{A}^n_{\mathbb{C}}$, $V/\langle g\rangle$ is nonsingular by [Che55, (A)]. Let $k > 1$ be minimal so that $h = g^k$ is a quasi-reflection. The quotient $V' = V/\langle h\rangle$ is nonsingular. Let $\mathrm{ord}(h) = \ell$ so $\mathrm{ord}(g{=}\ell k$. Let $\zeta_{\ell k}^{a_1}, \dots, \zeta_{\ell k}^{a_n}$ denote the eigenvalues of g acting on V so that $1, \dots, 1, \zeta_{\ell k}^{a_n k}$ are the eigenvalues of g^k. We consider the action of $\langle g\rangle/\langle h\rangle$ on V'. Since

$$V'/(\langle g\rangle/\langle h\rangle) \cong V/\langle g\rangle,$$

we can determine whether the singularities of $V/\langle g\rangle$ are canonical by applying the Reid–Tai criterion to $\langle g\rangle/\langle h\rangle$ acting on V', because in this case $\langle g\rangle/\langle h\rangle$ acts without quasi-reflections. The eigenvalues of $g^f\langle g^k\rangle$ acting on V' are $\zeta_{\ell k}^{fa_1}, \dots, \zeta_{\ell k}^{fa_{n-1}}, \zeta_{\ell k}^{\ell f a_n}$.

Therefore we define the modified Reid–Tai sum for roots of quasi-reflections when $\langle h\rangle$ contains all quasi-reflections in $\langle g\rangle$:

$$\textstyle\sum' \left(g^f\right) = \left\{\dfrac{fa_n}{k}\right\} + \displaystyle\sum_{j=1}^{n-1} \left\{\dfrac{fa_j}{\ell k}\right\}.$$

Let $H \subseteq \langle g\rangle$ denote the subgroup generated by all quasi-reflections in $\langle g\rangle$. Recall that $|H| \mid 6$ when considering ball quotients defined by the Eisenstein integers. If the quasi-reflection $h = g^k$ does not generate H, then it must be the case that $\mathrm{ord}(h) = 3$ and there exists another quasi-reflection $h' \in \langle g\rangle$ with order 2 (the order of $h = g^k$ is forced to be 3 by the minimality of k).

There is no quasi-reflection of order 6 by the minimality of k in the definition of $h = g^k$ and the assumption that H is not generated by a quasi-reflection. Note that h' must act nontrivially on a different coordinate than h, otherwise hh' would be a quasi-reflection of order 6. Without loss of generality, suppose that $h' = g^{k'}$ acts nontrivially on the $(n-1)$th coordinate of V, i.e., acts with eigenvalues $1, \ldots, \zeta_{\ell k}^{k' a_{n-1}}, 1$, where $\zeta_{\ell k}^{a_1}, \ldots, \zeta_{\ell k}^{a_{n-1}}, \zeta_{\ell k}^{a_n}$ are the eigenvalues of g acting on V. Now we have

$$V/\langle g\rangle \cong (V/H)/(\langle g\rangle/H),$$

where $\langle g\rangle/H$ acts on V/H without quasi-reflections. But we also have that

$$(V/H)/(\langle g\rangle/H) \cong (V'/\langle h'\langle h\rangle\rangle)/((\langle g\rangle/\langle h\rangle)/\langle h'\langle h\rangle\rangle),$$

where $V'/\langle h'\langle h\rangle\rangle$ is nonsingular since $\langle h'\langle h\rangle\rangle$ acts by quasi-reflection. The eigenvalues of $g^f\langle h'\langle h\rangle\rangle$ acting on $V'/\langle h'\langle h\rangle\rangle$ are $\zeta_{\ell k}^{f a_1}, \ldots, \zeta_{\ell k}^{\ell' f a_{n-1}}, \zeta_{\ell k}^{\ell f a_n}$, where ℓ' is the order of h' and ℓ is the order of h. From this we obtain another modified Reid–Tai sum:

$$\sum\nolimits'' \left(g^f\right) = \left\{\frac{f a_n}{k}\right\} + \left\{\frac{\ell' f a_{n-1}}{\ell k}\right\} + \sum_{j=1}^{n-2} \left\{\frac{f a_j}{\ell k}\right\}.$$

Note that in the setting of arithmetic ball quotients defined by the Eisenstein integers, we must have $\ell' = 2$ and $\ell = 3$.

The upshot of these modified Reid–Tai sums is summarized in the following lemma.

Lemma 4.4.6. *The arithmetic ball quotient $\mathbb{B}^n_\Lambda/\Gamma$ defined by the Eisenstein integers has canonical singularities if,*

(i) $\sum(g) \geq 1$ *for all $g \in \Gamma$ no power of which is a quasi-reflection,*

(ii) $\sum' \left(g^f\right) \geq 1$ *for $1 \leq f \leq k$, if $h = g^k$ is a quasi-reflection generating all quasi-reflections in $\langle g\rangle$, and*

(iii) $\sum'' \left(g^f\right) \geq 1$ *for f, if the subgroup $H \subseteq \langle g\rangle$ generated by quasi-reflections is not generated by a quasi-reflection.*

Proof. Each case corresponds to applying the Reid–Tai criterion to $\langle g\rangle/H$ acting on $T_{[\omega]}\mathbb{B}^n_\Lambda/H$. $\square$

Moreover, since $V/\langle g\rangle \cong V'/(\langle g\rangle/\langle h\rangle)$, if $\sum'\left(g^f\right) < 1$ we can still analyze what types of noncanonical singularities arise for $V/\langle g\rangle$ by examining the eigenvalues of $g^f\langle h\rangle$ acting on V'. This argument also applies if $\sum''(g^f) < 1$ via examining the eigenvalues of $g^f\langle h'\langle h\rangle\rangle$ acting on $V'/\langle h'\langle h\rangle\rangle$.

Remark 4.4.7. Note that this is distinct from [GHS07, Lemma 2.14] and [Beh11, Lemma 8] because we cannot guarantee that $\langle h\rangle$ generates all quasi-reflections in $\langle g\rangle$ for our setting.

Example 4.4.8. Let $\mathbb{Z}/6\mathbb{Z} = \langle g\rangle$ act on $\mathbb{C}^2$ via $(x,y)\mapsto(\zeta_6 x, \zeta_3^2 y)$. We compute that $\sum(g) = \frac{1}{6}+\frac{4}{6} < 1$ and $\sum(g^2) = \frac{1}{3}+\frac{1}{3} < 1$. But g^3 acts via $(x,y)\mapsto(-x,y)$, so g^3 is a reflection! So we cannot make any conclusions via Theorem 4.1.1.

We will show that $X = \mathbb{C}^2/\langle g\rangle$ has canonical singularities in two ways, first by computing the ring of invariants $\mathbb{C}[x,y]^{\langle g\rangle}$ and second by computing $\sum'(g^f)$. By an elementary calculation, $\mathbb{C}[x,y]^{\langle g\rangle} = \mathbb{C}[x^6, y^3, x^2y] \cong \mathbb{C}[X,Y,Z]/(XY - Z^3)$. Therefore $X = \operatorname{Spec}\mathbb{C}[x,y]^{\langle g\rangle}$ is an A_2-surface singularity, which is known to be canonical (Du Val singularities are canonical, see for instance [Dur79] or [Rei96, Chapter 4]).

By computing the modified Reid–Tai sum we may also show that X has canonical singularities. Indeed, $\sum'(g) = \frac{2}{6}+\frac{4}{6} = 1$ and $\sum'(g^2) = \frac{2}{3}+\frac{1}{3}$.

Example 4.4.9. This example shows how understanding the eigenvalues of $g\langle h\rangle$ on V' allows us to classify the cyclic quotient singularities $V/\langle g\rangle$ when some power of g is a quasi-reflection.

Let $\mathbb{Z}/6\mathbb{Z} = \langle g\rangle$ act on $\mathbb{C}^2$ via $(x,y)\mapsto(\zeta_6 x, \zeta_3 y)$. We see that $h = g^3$ acts as the reflection $(-x,y)$. Therefore, $V' = V/\langle h\rangle = \operatorname{Spec}(\mathbb{C}[x^2,y]) \cong \mathbb{A}^2_{\mathbb{C}}$. On these coordinates for V', $g\langle h\rangle$ acts via $(x^2,y)\mapsto(\zeta_3 x^2, \zeta_3 y)$. Therefore, $V/\langle g\rangle$ is a type $\frac{1}{3}(1,1)$ singularity. Indeed, $\sum'(g) = 2\cdot\frac{1}{6}+\frac{1}{3}$.

The next lemma will be used as a technical tool for studying most of the eigenvalues of $\langle g\rangle/H$ on V/H, in particular the eigenvalues contributing $\displaystyle\sum_{j=1}^{n-1}\left\{\frac{fa_j}{\ell k}\right\}$ to $\sum'(g)$ or $\displaystyle\sum_{j=1}^{n-2}\left\{\frac{fa_j}{\ell k}\right\}$ to $\sum''(g)$.

Lemma 4.4.10. *If $h = g^k$ is a quasi-reflection, then $\langle g\rangle/\langle h\rangle$ is a subgroup of $U(\Lambda')$ where Λ' is a sublattice of Λ such that the restriction of the Hermitian form on Λ' has signature $(1, n-1)$. In* *fact,* $\Lambda' = \delta^{\perp}$ *where* δ *is the primitive generator of the eigenvector of* g *corresponding to* ζ^{a_n}*, i.e., corresponding to the nontrivial eigenvector of* h*.*

Proof. This lemma follows from the details of the proof of [Beh11, Proposition 3]. We include the details for completeness.

Consider the decomposition of Λ_K as an h-module. Since h is a quasi-reflection, there must be exactly one distinguished eigenvalue λ on $\Lambda_{\mathbb{C}}$. This λ will appear on exactly one $V_d^{(k)}$ with multiplicity 1, therefore $\dim V_d^{(k)} = 1$ and, in fact, the λ-eigenspace of Λ_K is equal to $V_d^{(k)}$.

Let v be an eigenvector of g corresponding to the eigenvalue ζ^{a_n}. Then v comes from $V_d^{(k)}$ as it corresponds to the distinguished eigenspace of h. Since $\dim V_d^{(k)} = 1$, we can take $\delta \in \Lambda$ as a primitive generator of $V_d^{(k)} \cap \Lambda$.

We will show first that $(\delta, \delta) < 0$. Note that we usually use $h(\cdot, \cdot)$ to denote the Hermitian pairing of signature $(1, n)$. In this proof we omit h to avoid confusion with the quasi-reflection h. Let $\mathbb{W} = \mathrm{span}_{\mathbb{C}}(\omega)$, where $[\omega] \in \mathbb{B}^n_\Lambda$ is the fixed point we are considering of g. Since $[\omega]$ is fixed and h acts as a quasi-reflection, we have that $h\omega = \mu\omega$. Note that $\mu \neq \lambda$ because otherwise μ would be the distinguished eigenvalue of h and so the eigenvalues of h acting on $T_{[\omega]}\mathbb{B}^n_\Lambda$ would be $\mu^{-1}\lambda', \cdots, \mu^{-1}\lambda'$, where λ' is the other eigenvalue of h acting on $\Lambda_{\mathbb{C}}$. In particular, if $\mu = \lambda$, then h does not act as a quasi-reflection on $T_{[\omega]}\mathbb{B}^n_\Lambda$. Now we compute,

$$\begin{aligned}(\delta, \omega) &= (h\delta, h\omega)\\ &= (\lambda\delta, \mu\omega)\\ &= \lambda\overline{\mu}(\delta, \omega).\end{aligned}$$

If $(\delta, \omega) \neq 0$, then $\lambda\overline{\mu} = 1$ but since $|\mu| = 1$, this implies that $\lambda = \mu$, which contradicts the fact that h acts as a quasi-reflection on $T_{[\omega]}\mathbb{B}^n_\Lambda$. Therefore, $V_d^{(k)} \subset \mathbb{W}^{\perp}$. Since $[\omega] \in \mathbb{B}^n_\Lambda$, $(\cdot, \cdot)$ is positive definite on $\mathbb{W}$. Therefore $(\cdot, \cdot)$ is negative definite on $\mathbb{W}^{\perp}$ and hence $V_d^{(k)}$.

We then define the sublattice $\Lambda' \subset \Lambda$ via $\Lambda' = \delta^{\perp}$. Since $(\delta, \delta) < 0$, we have that Λ' is a lattice of signature $(n-1, 1)$ with Hermitian form given by $(\cdot, \cdot)|_{\Lambda'}$.

We can consider $[\omega] \in \mathbb{B}^{n-1}_{\Lambda'}$ since $\mathbb{W} \subseteq \Lambda'_{\mathbb{C}} \subseteq \Lambda_{\mathbb{C}}$. The cyclic group $\langle g \rangle$ acts as a subgroup of $U(\Lambda')$. Since the eigenvalues of h on Λ' are all μ, we have that h acts trivially on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$. Therefore $\langle g \rangle / \langle h \rangle \subseteq U(\Lambda')$. $\square$

Proposition 4.4.11. *Suppose $g \in \mathrm{Stab}_{\Gamma}([\omega])$, g^k acts as a quasi-reflection, and $n \geq 9$. Then the singularities of $\mathbb{B}^n_{\Lambda}/\langle g \rangle$ are either at worst canonical or of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$.*

Proof. Let $H \subseteq \langle g \rangle$ be the subgroup generated by quasi-reflections and let $V = T_{[\omega]}\mathbb{B}^n_{\Lambda}$. We want to determine the eigenvalues of gH acting on V/H. As previously discussed we have two cases, either $g^k = h$ generates H or H is generated by quasi-reflections h and h' of order 3 and 2.

We first assume that $H = \langle h \rangle$. Let $V' = V/\langle g^k \rangle$. We then have $V/\langle g \rangle \cong V'/(\langle g \rangle / \langle h \rangle)$. In particular, if $\zeta_{k\ell}^{a_1}, \dots, \zeta_{k\ell}^{a_n}$ are the eigenvalues of g acting on $T_{[\omega]}\mathbb{B}^n_{\Lambda}$ with $\zeta_{k\ell}^{a_n}$ corresponding to the distinguished eigenvalue of h, then

$$\sum{}'(g) = \left\{\frac{a_n}{k}\right\} + \sum_{j=1}^{n-1} \left\{\frac{a_j}{k\ell}\right\}.$$

Since $n-1 \geq 7$, Lemma 4.4.10 allows us to study $\displaystyle\sum_{j=1}^{n-1} \left\{\frac{a_j}{k\ell}\right\}$ by studying the action of $\langle g \rangle / \langle h \rangle$ on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$. We now have two cases, either $\langle g \rangle / \langle h \rangle$ on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$ acts without quasi-reflections or acts with quasi-reflections.

First assume that $\langle g \rangle / \langle h \rangle$ acts on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$ without quasi-reflections. Note that $\sum(g\langle h \rangle)$ on $\mathbb{B}^{n-1}_{\Lambda'}$ is equal to $\sum_{j=1}^{n-1} \left\{\frac{a_j}{k\ell}\right\}$. Therefore, either $\sum(g^f\langle h \rangle) \geq 1$ for $1 \leq f < k$, which implies that $V'/(\langle g \rangle / \langle g^k \rangle) \cong V/\langle g \rangle$ has a canonical singularity at $\overline{\omega}$ or $\mathbb{B}^{n-1}_{\Lambda'}/(\langle g \rangle / \langle h \rangle)$ has a quotient singularity from the list in Proposition 4.3.12.

We now consider the possible values of $\left\{\frac{a_n}{k}\right\}$. We can interpret this quantity as taking the contribution to $\sum(g)$ of the eigenvalue on the tangent space $\zeta_{\ell k}^{a_n}$, multiplying by ℓ, and then taking the fractional part of this result. By Proposition 4.4.5, the eigenvalue $\zeta_{\ell k}^{a_n}$ is equal to $\lambda^{-1}\mu$ where λ is the eigenvalue of g acting on $\mathbb{W}$ and $\mu \in \mathcal{O}^{\times}$. Note that if $|\lambda| \notin \{1,2,3,4,6,12\}$, then $\sum(g^f\langle h \rangle) \geq 1$ on $\mathbb{B}^{n-1}_{\Lambda'}$ which implies $\sum'(g^f) \geq 1$ by Lemma 4.3.4. So we can assume that

$|\lambda| \in \{1,2,3,4,6,12\}$. By considering all possible rth roots of unity λ^{-1} and all $\mu \in \mathcal{O}^\times$, we have that $\left\{\frac{a_n}{\ell k}\right\} \in \{k/12 : 0 \leq k < 12\}$. Indeed, products of 12th roots of unity and 6th roots of unity are 12th roots of unity. Since $\ell \in \{2,3,6\}$, we have that $\left\{\frac{a_n}{k}\right\} \in \{k/6 : 0 \leq k < 6\}$.

By considering all possible singularities from the list in Proposition 4.3.12, we see that by adding this contribution we will always either force $\sum'(g) \geq 1$ or obtain another singularity already on the list, except in the case where the singularity on $\mathbb{B}^{n-1}_{\Lambda'}$ is of type $\frac{1}{3}(1,1)$ and $\left\{\frac{a_n}{k}\right\}$ contributes 1/6. Here we appeal to the following argument. We can only obtain a contribution of 1/6 to $\left\{\frac{a_n}{k}\right\}$ if the contribution of $\lambda^{-1}\mu$ is 1/12. This only happens when λ^{-1} is order 4 or 12. But then we cannot obtain a singularity of type $\frac{1}{3}(1,1)$ on $\mathbb{B}^{n-1}_{\Lambda'}$ by Proposition 4.3.14, so this case is impossible.

Now we assume that $\langle g\rangle/\langle h\rangle$ acts on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$ with quasi-reflections. Note that $\langle g\rangle/\langle h\rangle$ acts on $V' = T_{[\omega]}\mathbb{B}^n_\Lambda/\langle h\rangle$ without quasi-reflections by our assumption that $\langle h\rangle$ is the subgroup of $\langle g\rangle$ generated by quasi-reflections. Therefore, we can apply the Reid–Tai criterion directly to elements in $\langle g\rangle/\langle h\rangle$ acting on V'. If $g^f\langle h\rangle$ acts as a quasi-reflection on $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$, then by Lemma 4.4.10 the eigenvalues of $g^f\langle h\rangle$ on V' are given by $1,\ldots,1,\zeta_6^a,\zeta_6^{\ell b}$, where ℓ is the order of h. Note that no power $(g^f)^m$ can act as a quasi-reflection on V' by assumption. This restriction leads to the following possible cases for each $\ell \in \{2,3,6\}$.

If $\ell = 2$, then $\zeta_6^{\ell b} = \zeta_3^b$. Therefore, ζ_6^a must have order 3 so that $(g^f)^m$ acts without quasi-reflections. So we have that either $a = 2, b = 1$, corresponding to the case of $\frac{1}{3}(1,1)$, or $a = 4, b = 1$, causing to $\sum'(g^f) \geq 1$, or $a = 2$, $b = 2$, also causing $\sum'(g^f) \geq 1$, or $a = 4$, $b = 2$, also causing $\sum'(g^f) \geq 1$. We will reference these four cases later.

If $\ell = 3$, then $\zeta_6^{\ell b} = (-1)^b$. So the only option that doesn't lead to $g^f\langle h\rangle$ acting as a quasi-reflection on V' is $a = 3, b = 1$, but this yields $\sum'(g^f) \geq 1$. We will reference this case later.

The case $\ell = 6$ causes $g^f\langle h\rangle$ to act as a quasi-reflection on $V' \Rightarrow g^f \in \langle h\rangle$.

Now we consider the case where $\langle h\rangle$ is not equal to the subgroup generated by quasi-reflections $H \subseteq \langle g\rangle$. In this case, we follow the strategy given by Lemma 4.4.6. We apply Lemma 4.4.10 twice; first to h as before in order to obtain Λ', second to $h'\langle h\rangle$ acting as a quasi-reflection on $T_{[\omega]}\mathbb{B}^n_\Lambda$ and hence $T_{[\omega]}\mathbb{B}^{n-1}_{\Lambda'}$ in order to obtain a sublattice $\Lambda'' \subseteq \Lambda'$ of signature $(1, n-2)$ on which $\langle g\rangle/H$ is a subgroup of $\mathrm{U}(\Lambda'')$. As in the previous case, we now have two cases. Either $\langle g\rangle/H$ acts on $T_{[\omega]}\mathbb{B}^{n-2}_{\Lambda''}$ without quasi-reflections or acts with quasi-reflections.

If we assume that $\langle g\rangle/H$ acts on $T_{[\omega]}\mathbb{B}^{n-2}_{\Lambda''}$ without quasi-reflections, then we apply Proposition 4.3.12 since $n-2\geq 7$. By the same argument as the previous case, we obtain the same list of possible singularities.

Otherwise, by Lemma 4.4.10 the eigenvalues of $g^f H$ on $T_{[\omega]}\mathbb{B}^n_\Lambda/H$ are given by $1, \ldots, 1, \zeta_6^a$, ζ_6^{2b}, ζ_6^{3c}. As in the previous case, no power of $g^f H$ can act on $T_{[\omega]}\mathbb{B}^n_\Lambda/H$ as a quasi-reflection. This restricts us to the following cases.

If all $a,b,c\neq 0$, then in all cases we have $\sum''(g^f)\geq 1$ (the minimal case is $a=b=c=1$).

If $c=0$, then we obtain the four cases corresponding to $\ell=2$ from the case of $H=\langle h\rangle$.

If $b=0$, then we obtain the case corresponding to $\ell=3$ from the case of $H=\langle h\rangle$.

If $a=0$, then $(g^f H)^2$ acts as a quasi-reflection, so $g^f\in H$ by the definition of H. □

Theorem 4.4.12. *If $n\geq 9$ and $\mathbb{B}^n_\Lambda$ is defined by the Eisenstein integers, then then the singularities of $\mathbb{B}^n_\Lambda/\Gamma$ are either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g\rangle\subseteq G$ so that $\mathbb{C}^n/\langle g\rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$.*

Proof. Proposition 4.3.12 and Proposition 4.4.11. □

Corollary 4.4.13. *In the setting of Theorem 4.4.12, if $\sum(g)<1$ there exists a 1-dimensional $V_d^{(k)}$ contributing a nontrivial eigenvalue to the action of g on $T_{[\omega]}\mathbb{B}^n_\Lambda$.*

Proof. Corollary 4.3.13 and consider all decompositions appearing in the proof of Proposition 4.4.11 leading to $\sum(g)<1$. □

Remark 4.4.14. The existence of noncanonical cyclic quotient singularities does not imply that $\mathbb{B}^n_\Lambda/\Gamma$ has noncanonical singularities due to the existence of quasi-reflections in $\mathrm{Stab}_\Gamma([\omega])$. Due to [Tai82, Proposition 3.1], this result implies that if $\mathbb{B}^n_\Lambda/\Gamma$ has a noncanonical singularity at $\overline{[\omega]}$, then there exists some $g \in \mathrm{Stab}_\Gamma([\omega])$ such that $\mathbb{B}^n_\Lambda/\langle g \rangle$ is noncanonical and of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$, if $n \geq 8$.

Chapter 5

Singularities at the Toroidal Boundary

In this chapter we study the singularities on the boundary of the toroidal compactification $\overline{\mathbb{B}^n_\Lambda/\Gamma}$. We first recall some background on toroidal compactifications [AMRT10]. In the general setting, there are choices involved in constructing the toroidal compactification that lead to different compactifications. For arithmetic ball quotients, these choices no longer matter and so there is a unique toroidal compactification. It is projective and smooth up to finite quotient singularities.

We construct the toroidal compactification by considering partial compactifications of $\mathbb{B}^n_\Lambda/\Gamma$ near each cusp appearing in the Satake–Baily–Borel compactification. The cusps are in bijection with the isotropic subspaces of Λ_K modulo Γ. We construct $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ by performing partial compactifications at each cusp and then gluing these spaces back together. In order to study singularities, it suffices to understand the singularities appearing on the partial compactifications. We will show that the singularities appearing at the boundary of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ are controlled by the representation theory over K by choosing a convenient basis for Λ_K.

5.1 Local Coordinates for $\mathbb{B}^n_\Lambda(F)$

Let $\ell_K \subseteq \Lambda_K$ be an isotropic subspace with respect to the Hermitian form $h(\cdot,\cdot)$. Note that ℓ_K is 1-dimensional because h has signature $(1,n)$. We prove the following lemma which gives a nice basis for computations.

Notation: $A^* = \overline{A}^T$.

Lemma 5.1.1. *There exists a basis $b_1,\ldots,b_{n+1}$ of Λ_K, such that* *(i)* b_1 *is a basis of* ℓ_K *and* $b_1, \dots, b_n$ *is a basis of* $\ell_K^\perp$*,*

(ii) the Hermitian form is given with respect to this basis as

$$Q := (h(b_i, b_j))_{1 \le i,j \le n} = \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & 0 \\ \hline \overline{a} & 0 & 0 \end{array}\right),$$

where $a \in K$ *and* $B = B^*$*.*

Moreover, we may choose $b_1, \dots, b_{n+1}$ *such that they correspond to generators of* Λ *over* $\mathcal{O}$ *via extension of scalars. In this case,* a *and the entries of* B *are given by elements of* $\mathcal{O}$*.*

Proof. See [Beh11, Lemma 9] or [GHS07, Lemma 2.24] for similar proofs.

Choose a basis $\{e_1, \dots, e_{n+1}\}$ for Λ_K such that $\ell_K = \operatorname{span}_K(e_1)$ and $\ell_K^\perp = \operatorname{span}_K(e_1, \dots, e_n)$. Let Q' be the matrix for h with respect to this basis. Note that Q' must be of the form

$$Q' = \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & c \\ \hline \overline{a} & c^* & d \end{array}\right).$$

Indeed, the upper zeroes follow from that fact that $h(\ell_K, e) = 0$ for all $e \in \ell_K^\perp$. The rest of the entries are determined by the fact that $Q' = Q'^*$.

Note that B has non-zero determinant, because it represents the Hermitian form h on $\ell_K^\perp/\ell_K$. So B^{-1} exists. Define

$$N := \left(\begin{array}{c|c|c} 1 & 0 & r \\ \hline 0 & I_{n-1} & R \\ \hline 0 & 0 & 1 \end{array}\right),$$

where $R := -B^{-1}c \in \mathbb{Q}(\zeta_3)^{n-1}$ and we choose $r \in \mathbb{C}$ such that it satisfies the equation

$$d - c^*B^{-1}c + \overline{r}a + \overline{a}r = 0.$$

This is possible since $\overline{d} = d \in \mathbb{R}$, $c^*B^{-1}c = h'(c, c) \in \mathbb{R}$, and the last two terms in the sum are complex conjugates.

Then we compute

$$N^*Q'N = \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & BR+c \\ \hline \overline{a} & R^*B+c^* & \delta \end{array}\right),$$

where $\delta = \overline{a}r + (R^*B + c^*)R + \overline{r}a + R^*c + d$. Notice that by the definition of R we have

$$BR + c = B(-B^{-1}c) + c = 0.$$

Using the fact that $(B^{-1})^* = B^{-1}$ and the definition of r, we also have

$$\begin{aligned}\delta &= \overline{a}r + ((-B^{-1}c)^*B + c^*)(-B^{-1}c) + \overline{r}a + (-B^{-1}c)^*c + d \\ &= \overline{a}r + (-c^*B^{-1}B + c^*)(-B^{-1}c) + \overline{r}a + (-c^*B^{-1})c + d \\ &= \overline{a}r + 0 + \overline{r}a - c^*B^{-1}c + d \\ &= 0.\end{aligned}$$

The final statement in the lemma follows from the fact that K is the fraction field of $\mathcal{O}$ and so we can rescale our basis to become generators of Λ by 'clearing denominators'. □

Let F be the cusp corresponding to the isotropic subspace ℓ_K. In order to construct the toroidal compactification, we need to find the following groups;

- $N(F) \subseteq \Gamma_{\mathbb{R}}$, the stabilizer subgroup corresponding to F,
- $W(F) \subseteq N(F)$, the unipotent radical of $N(F)$, i.e., the largest normal subgroup of $N(F)$ consisting only of elements X such that $(X - I)^n = 0$ for some n,
- $U(F) \subseteq W(F)$, the center of $W(F)$,
- $U(F)_{\mathbb{Z}} = U(F) \cap \Gamma$, the integral elements of the center.

The first two groups are easy to compute explicitly with respect to our basis from the previous lemma.

Lemma 5.1.2. *The stabilizer $N(F)$ is given by*

$$N(F) = \left\{ g = \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) \in \Gamma_{\mathbb{R}} : \begin{array}{c} z\overline{u} = 1,\ X^*BX = B, \\ X^*By + v^*az = 0, \\ y^*By + \overline{za}w + za\overline{w} = 0 \end{array} \right\}. \tag{5.1.1}$$

Proof. See [Beh11, Lemma 4.2]. We include the details for completeness.

Let

$$g = \left(\begin{array}{c|c|c} u & v & w \\ \hline e & X & y \\ \hline f & m & z \end{array}\right) \in N(F).$$

By definition, g must stabilize $\ell_{\mathbb{R}}$, so $gb_1 \in \ell_{\mathbb{R}}$. Hence,

$$\left(\begin{array}{c|c|c} u & v & w \\ \hline e & X & y \\ \hline f & m & z \end{array}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ e \\ f \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix}.$$

Therefore $e = f = 0$. Moreover, we have that $N(F) \subseteq \Gamma_{\mathbb{R}} \subseteq U(\Lambda)$. So we have that $g^*Qg = Q$. We compute

$$\begin{aligned} g^*Qg &= \left(\begin{array}{c|c|c} \overline{u} & 0 & 0 \\ \hline v^* & X^* & m^* \\ \hline \overline{w} & y^* & \overline{z} \end{array}\right) \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & 0 \\ \hline \overline{a} & 0 & 0 \end{array}\right) \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & m & z \end{array}\right) \\ &= \left(\begin{array}{c|c|c} 0 & 0 & a\overline{u} \\ \hline 0 & X^*B & av^* \\ \hline \overline{az} & y^*B & \overline{w}a \end{array}\right) \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & m & z \end{array}\right). \end{aligned}$$

Note that the 2nd entry in the first row will be $a\overline{u}m$. Since $g^*Qg = Q$, we have that $m = 0$. So we have that

$$g^*Qg = \left(\begin{array}{c|c|c} 0 & 0 & a\overline{u}z \\ \hline 0 & X^*BX & X^*By + azv^* \\ \hline \overline{az}u & \overline{az}v + y^*BX & \overline{az}w + y^*By + \overline{w}az \end{array}\right).$$

All of our conditions come from the fact that this matrix is equal to Q. $\square$

Lemma 5.1.3. *The unipotent radical of $N(F)$ is*

$$W(F) = \left\{ g = \left(\begin{array}{c|c|c} 1 & v & w \\ \hline 0 & I_{n-1} & y \\ \hline 0 & 0 & 1 \end{array}\right) : \begin{array}{c} By + v^*a = 0, \\ y^*By + \overline{a}w + a\overline{w} = 0 \end{array} \right\}. \tag{5.1.2}$$

Proof. [Beh11, Lemma 11]. □

Lemma 5.1.4. *The center of $W(F)$ is given by*

$$U(F) = \left\{ g = \left(\begin{array}{c|c|c} 1 & 0 & iax \\ \hline 0 & I_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) : x \in \mathbb{R} \right\}. \tag{5.1.3}$$

Proof. We follow [Beh11, Lemma 12]. Let

$$g = \left(\begin{array}{c|c|c} 1 & v & w \\ \hline 0 & I_{n-1} & y \\ \hline 0 & 0 & 1 \end{array}\right), \quad g' = \left(\begin{array}{c|c|c} 1 & v' & w' \\ \hline 0 & I_{n-1} & y' \\ \hline 0 & 0 & 1 \end{array}\right) \in W(F).$$

Then

$$gg' = \left(\begin{array}{c|c|c} 1 & v' + v & w' + vy' + w \\ \hline 0 & I_{n-1} & y + y' \\ \hline 0 & 0 & 1 \end{array}\right)$$

and

$$g'g = \left(\begin{array}{c|c|c} 1 & v' + v & w' + v'y + w \\ \hline 0 & I_{n-1} & y + y' \\ \hline 0 & 0 & 1 \end{array}\right).$$

So $g \in U(F)$ if and only if $gg' = g'g$ for all $g' \in W(F)$ if and only if $vy' = v'y$ for all v', y' as in Lemma 5.1.3. Since $g \in W(F)$, $v = (-a^{-1}By)^*$. This leads to

$$vy' = v'y$$

$$(-a^{-1}By)^*y' = (-a^{-1}By')^*y$$

$$y^*By' = y'^*By$$
$$y^*By' - (y^*By')^* = 0$$

This implies that $y^*By' \in \mathbb{R}$ for every y'. Since B is invertible and every $y' \in \mathbb{C}^{n-1}$ appears for some element of $W(F)$, we can rephrase this as $y^*z' \in \mathbb{R}$ for all $z' \in \mathbb{C}^{n-1}$. Let $z' = e_j$, the standard unit vector in the jth component with respect to our fixed basis. Then $y^*e_j \in \mathbb{R} \Rightarrow \overline{y_j} \in \mathbb{R}$, where y_j is the jth component of y. Now let $z' = ie_j$. Then $y^*ie_j = \overline{y_j} \cdot i \in \mathbb{R} \Rightarrow y_j \in i\mathbb{R}$. Therefore $y_j \in \mathbb{R} \cap i\mathbb{R} = \{0\}$ for all j. So, $y = 0$. By Lemma 5.1.3, $g \in W(F) \Rightarrow By + v^*a = 0$. But $y = 0$, so $v^*a = 0$. Since $a \neq 0$, we have that $v^* = 0 \Rightarrow v = 0$.

Now we want to investigate w using the condition $\overline{a}w + \overline{w}a = 0$. Since $\overline{a}w = -\overline{\overline{a}w}$, we have that $\overline{a}w \in i\mathbb{R}$. But then we have that

$$\overline{a}w \in i\mathbb{R} \Leftrightarrow \overline{a}iw \in \mathbb{R} \Leftrightarrow iw \in a\mathbb{R}.$$

Therefore, $w \in ia\mathbb{R}$. $\square$

Now we will specialize to the case $K = \mathbb{Q}(\zeta_3)$.

Lemma 5.1.5. *In the case $K = \mathbb{Q}(\zeta_3)$, $U(F)_{\mathbb{Z}} = U(F) \cap \Gamma \cong \mathbb{Z}$. This isomorphism is given by*

$$iax = \sigma b \mapsto b,$$

where $\sigma = \frac{ia \ \mathrm{lcm}(sp,3rq)}{6rp}\sqrt{3}$ with $a = e + f\sqrt{-3}$, $e = \frac{p}{q}$, and $f = \frac{s}{r}$ in reduced form.

Proof. We apply [Beh11, Lemma 13] to the case $D = -3$. See the proof of the lemma for the details concerning the definition of σ. $\square$

For example, when $a = 1$, we have that all numbers of the form $iax = ix$ where $x \in \mathbb{R}$ and $iax \in \mathbb{Z}[\zeta_3]$ are generated by $i\sqrt{3}b$ with $b \in \mathbb{Z}$, so $\sigma = i\sqrt{3}$.

In order to compute the toroidal compactification, we first need to compute the partial compactification in the direction of F. We do this by defining $\mathbb{B}^n_\Lambda(F) = \mathbb{B}^n_\Lambda / U(F)_{\mathbb{Z}}$. Since $U(F)_{\mathbb{Z}} \cong$

$\mathbb{Z}$ and it acts on $\mathbb{B}^n_\Lambda$ in local coordinates $[t_1 : t_2 : \cdots : t_n : t_{n+1}] = [\alpha : t : 1]$ with respect to the basis found in Lemma 5.1.1 by

$$\begin{pmatrix} 1 & 0 & \sigma b \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha + \sigma b \\ t \\ 1 \end{pmatrix},$$

we can identify $\mathbb{B}^n_\Lambda(F)$ locally with $\mathbb{C}^* \times \mathbb{C}^{n-1}$ with coordinates given by (α, t) (using $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$). Note that we can take $t_{n+1} = 1$ in these coordinates because if $t_{n+1} = 0$, then by Lemma 5.1.1 $h((t_1, \ldots, t_n, t_{n+1}), (t_1, \ldots, t_n, t_{n+1})) = a\overline{t_1}t_{n+1} + \overline{a}t_1\overline{t_{n+1}} + t^*Bt = t^*Bt$. But h on $\operatorname{span}_{\mathbb{C}}\{b_1, b_{n+1}\}$ has signature $(1,1)$, so $t^*Bt < 0$, thus $[t_1, \ldots, t_n, t_{n+1}] \notin \mathbb{B}^n_\Lambda$.

We want to be sure to find coordinates so that the action of the group $G(F) = N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$ is equivariant. Therefore, we are motivated to define the new coordinate θ on $\mathbb{C}^*$:

$$\theta = e^{\frac{2\pi i}{\sigma}\alpha}.$$

Now we have that $g \cdot \theta = e^{\frac{2\pi i}{\sigma}(\alpha+\sigma b)} = e^{\frac{2\pi i}{\sigma}\alpha} = \theta$.

In these new coordinates (θ, t) on $\mathbb{C}^* \times \mathbb{C}^{n-1}$, the action of $G(F)$ uniquely extends to the locus where $\theta = 0$. Thus we can locally compactify in the direction of F by allowing $\theta = 0$. The boundary is given by all points of the form $(0, t)$. We denote this partial compactification by $\overline{\mathbb{B}^n_\Lambda(F)}$.

We will now compute the Reid–Tai sum for points on the boundary. An element of Γ stabilizing a point on the boundary in the direction of F must stabilize F, so is an element of $N(F)_{\mathbb{Z}}$. Therefore it suffices to check the Reid–Tai sum on the partial quotient we have constructed, i.e., examine the eigenvalues of elements $g \in G(F)$ acting on $\overline{\mathbb{B}^n_\Lambda(F)}$.

Suppose that some element $g \in G(F)$ stabilizes a point $\omega = (0, t)$ in the boundary. We want to compute $\sum(g)$. In order to do this, we show how g acts on the tangent space to $\overline{\mathbb{B}^n_\Lambda(F)}$ at ω. In a complex analytic neighborhood, we have that $\overline{\mathbb{B}^n_\Lambda(F)} \cong \mathbb{C}^n$, therefore the Jacobian gives the action on the tangent space.

If $g \in N(F)_{\mathbb{Z}}$, we have shown that

$$g = \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right)$$

with all entries in $\mathbb{Z}[\zeta_3]$, $z\overline{u} = 1$, $X^*BX = B$, $X^*By + v^*az = 0, y^*By + \overline{za}w + za\overline{w} = 0$.

We can then compute that $g \in N(F)_{\mathbb{Z}}$ acts on $\mathbb{B}^n_\Lambda$ via

$$\begin{aligned} g \cdot \begin{pmatrix} \alpha \\ t \\ 1 \end{pmatrix} &= \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) \begin{pmatrix} \alpha \\ t \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} u\alpha + vt + w \\ Xt + y \\ z \end{pmatrix} \\ &= z \begin{pmatrix} z^{-1}(u\alpha + vt + w) \\ z^{-1}(Xt + y) \\ 1 \end{pmatrix} \end{aligned}$$

and thus acts on the local coordinates $(\alpha, t) \in \mathbb{C} \times \mathbb{C}^{n-1}$ via

$$\alpha \mapsto z^{-1}(u\alpha + vt + w)$$

$$t \mapsto z^{-1}(Xt + y).$$

But we need the action on the local coordinates $(\theta, t) \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. We only need to remember that $\theta = e^{\frac{2\pi i}{\sigma}\alpha}$. So we have that $g \cdot \theta = e^{\frac{2\pi i}{\sigma} z^{-1}(u\alpha + Vt + w)} = e^{\frac{2\pi i}{\sigma} z^{-1} u\alpha} e^{\frac{2\pi i}{\sigma} z^{-1}(Vt+w)}$. But we have that $z \in \mathbb{Z}[\zeta_3]$ and $z\overline{u} = 1$, therefore $z = \pm 1, \pm\zeta_3, \pm\zeta_3^2$ and $u = z$. So the action of g on $(\theta, t) \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ is given by

$$\theta \mapsto \theta e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)}$$

$$t \mapsto z^{-1}(Xt + y).$$

Since these coordinates are invariant under the action of $U(F)_{\mathbb{Z}}$, the action extends to allow $\theta = 0$, so we get that $g \in N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$ acts on the partial compactification $\overline{\mathbb{B}^n_\Lambda(F)} = \mathbb{C}^n$ via

$$\theta \mapsto \theta e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)}$$

$$t \mapsto z^{-1}(Xt + y).$$

Suppose that a point on the boundary $(0, t)$ has nontrivial stabilizer. We remark that the quotient $\overline{\mathbb{B}^n_\Lambda(F)}/G(F)$ gives a local description of the toroidal compactification in the direction of the cusp F prior to gluing, therefore we use the Reid–Tai criterion on the coordinates we have specified above in order to describe the nature of the singularities on the boundary. Suppose $g \in G(F)$ stabilizes the boundary point $(0, t)$. We will apply the Reid–Tai criterion by studying the action of g on the tangent space to $\mathbb{C}^n$ at $(0, t)$. The tangent space to $\mathbb{C}^n$ is canonically isomorphic to $\mathbb{C}^n$ and the action is given by taking derivatives. So we have that the action on the tangent space at $(0, t)$ is given by the matrix

$$J = \begin{pmatrix} e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)} & 0 \\ * & z^{-1}X \end{pmatrix}.$$

The eigenvalues of J are given by $e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)}$ and the eigenvalues of $z^{-1}X$. In order to determine the eigenvalues of X, we want to utilize the fact that X acts as a representation on a $\mathbb{Q}(\zeta_3)$-vector space so that we can apply our result from representation theory.

Observe that the coordinate $t \in \mathbb{C}^{n-1}$ was chosen to be defined over $K = \mathbb{Q}(\zeta_3)$ by Lemma 5.1.1. Moreover, as a consequence of the same lemma, we have that the entries of $z^{-1}X$ are in K. Therefore, we can view g as acting on the K-vector space spanned by $b_2, \ldots, b_n$ via the matrix $z^{-1}X$ as giving an $(n-1)$-dimensional K-representation of $\langle g \rangle$.

5.2 Singularities on the Toroidal Boundary

From the previous discussion, if we can make sense of the eigenvalue $e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)}$, then we can completely classify the singularities on the boundary using results from Chapter 4 applied to viewing $z^{-1}X$ as a $\mathbb{Q}(\zeta_3)$-representation coming from the action of g. Following the notation of the proof of [Beh11, Proposition 4], let m_X be the order of the matrix X. As in Chapter 4, we will first assume that no power of g acts as a quasi-reflection on the fixed point $(0,t)$ in the boundary. We begin by imposing a restriction on m_X.

Lemma 5.2.1. *Suppose we have chosen coordinates as in Lemma 5.1.1 and that $\langle g\rangle \subset N(F)_{\mathbb{Z}}$ acts on the fixed point $(0,t)$ without quasi-reflections. If $m_X \notin \{1,2,3,4,6,12\}$, then $\sum(g) \geq 1$.*

Proof. Suppose $m_X \notin \{1,2,3,4,6,12\}$. Then X has an eigenvalue ζ_d over $\mathbb{C}$ such that ζ_d is a primitive dth root of unity and $d \notin \{1,2,3,4,6,12\}$. Indeed if all eigenvalues of X were primitive dth roots of unity with $d \in \{1,2,3,4,6,12\}$, then $m_X \in \{1,2,3,4,6,12\}$. Therefore X contains some $V_d^{(k)}$ with $d \notin \{1,2,3,4,6,12\}$. By Lemma 5.1.1, $z \in \mathcal{O}^\times$, so $z \in \{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$. Therefore, the eigenvalues of $z^{-1}V_d^{(k)}$ cause $\sum(g) \geq 1$ by Lemma 4.3.4. Indeed, the eigenvalues of $z^{-1}V_d^{(k)}$ can be viewed as eigenvalues of g on $T_{[\omega]}\mathbb{B}^n_\Lambda$ in the cases where the order of λ, the eigenvalue acting on $\mathbb{W}$, has order dividing 6, i.e., $r \in \{1,2,3,6\}$. $\square$

Lemma 5.2.2. *If $m_X \in \{1,2,3,6\}$, then* $\left(e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}\right)^6 = 1$.

Proof. Let

$$g = \left(\begin{array}{c|c|c} z & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) \in N(F)_{\mathbb{Z}}.$$

Note that $z \in \mathcal{O}^\times$, therefore

$$g^6 = \left(\begin{array}{c|c|c} 1 & V & 6z^5w + 5z^4vy + 4z^3vXy + 3z^2vX^2y + 2zvX^3y + vX^4y \\ \hline 0 & I & Y \\ \hline 0 & 0 & 1 \end{array}\right),$$

where $V = \displaystyle\sum_{j=0}^{5} z^j v X^{5-j}$ and $Y = \displaystyle\sum_{j=0}^{5} z^j X^{5-j} y$. Since g^6 fixes $(0,t)$, we have that $t = t + Y$, therefore $Y = 0$. But then $BY + V^*a = 0 \Rightarrow V^*a = 0 \Rightarrow V^* = 0 \Rightarrow V = 0$.

Since $g \in N(F)_{\mathbb{Z}}$, we have shown that $g^6 \in U(F)_{\mathbb{Z}}$. Therefore we obtain the relations

$$\sum_{j=0}^{5} z^j v X^{5-j} = 0,$$

$$\sum_{j=0}^{5} z^j X^{5-j} y = 0,$$

$$6z^5 w + 5z^4 vy + 4z^3 vXy + 3z^2 vX^2 y + 2zvX^3 y + vX^4 y \equiv 0 \pmod{\sigma}.$$

We want to show that $6z^{-1}(vt+w) \equiv 0 \pmod{\sigma}$. Since $(0,t)$ is a fixed point, we have that

$$t = z^{-1}(Xt + y),$$

and since $z \in \mathcal{O}^\times$, $z^{-1} = z^5$. Using all of the above relations, we compute

$$\begin{aligned}
6z^{-1}(vt+w) &= 6z^5 vt + 6z^5 w \\
&\equiv 6z^5 vt - 5z^4 vy - 4z^3 vXy - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + 5z^5 vt - 5z^4 vy - 4z^3 vXy - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + 5z^4 v(zt - y) - 4z^3 vXy - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + 5z^4 vXt - 4z^3 vXy - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + 4z^4 vXt - 4z^3 vXy - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + 4z^3 vX(zt - y) - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + 4z^3 vX^2 t - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + 3z^3 vX^2 t - 3z^2 vX^2 y - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + 3z^2 vX^2 (zt - y) - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + 3z^2 vX^3 t - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + z^2 vX^3 t + 2z^2 vX^3 t - 2zvX^3 y - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + z^2 vX^3 t + 2zvX^3 (zt - y) - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + z^2 vX^3 t + 2zvX^4 t - vX^4 y \\
&= z^5 vt + z^4 vXt + z^3 vX^2 t + z^2 vX^3 t + zvX^4 t + zvX^4 t - vX^4 y
\end{aligned}$$

$$
\begin{aligned}
&= z^5vt + z^4vXt + z^3vX^2t + z^2vX^3t + zvX^4t + vX^4(zt - y) \\
&= z^5vt + z^4vXt + z^3vX^2t + z^2vX^3t + zvX^4t + vX^5t \\
&= z^5vt + z^4vXt + z^3vX^2t + z^2vX^3t + zvX^4t + vX^5t \\
&= \left(\sum_{j=0}^{5} z^j vX^{5-j}\right) t \\
&= 0 \pmod{\sigma}.
\end{aligned}
$$

□

Lemma 5.2.3. *If $m_X \in \{4, 12\}$, then $\left(e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)}\right)^{12} = 1$.*

Proof. Similar to previous lemma, using the fact that

$$
g^{12} = \left(\begin{array}{c|c|c}
1 & \sum_{j=0}^{11} z^j vX^{11-j} & 12z^{11}w + \sum_{j=0}^{10}(j+1)z^j vX^{10-j} \\
\hline
0 & I & \sum_{j=0}^{11} z^j X^{11-j} y \\
\hline
0 & 0 & 1
\end{array}\right) \in U(F)_{\mathbb{Z}}
$$

□

Proposition 5.2.4. *Suppose $\mathbb{B}^n_\Lambda/\Gamma$ is defined by the Eisenstein integers, $n \geq 8$, and $G(F)$ does not contain quasi-reflections. Then either $\overline{\mathbb{B}^n_\Lambda(F)}/G(F)$ has canonical singularities , or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g \rangle \subseteq G$ so that $\mathbb{C}^n/\langle g \rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, $\frac{1}{12}(1,2,7)$, or $\frac{1}{12}(1,1,2,7)$.*

Proof. By Lemma 5.2.1, we restrict to the case of $m_X \in \{1,2,3,4,6,12\}$ and $z \in \mathcal{O}^\times$, which corresponds to the cases of $r \in \{1,2,3,6\}$ of Proposition 4.3.12. Note that in the decomposition of X as a K-representation, if $V_d^{(k)}$ appears with $d \notin \{1,2,3,4,6,12\}$, then $\sum(g) \geq 1$ by Lemma 4.3.4. If $e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)} = 1$, then we obtain all possible singularities from the list in Proposition 4.3.12. Therefore we can assume $e^{\frac{2\pi i}{\sigma} z^{-1}(vt+w)} \neq 1$. We proceed in cases depending on the value of m_X.

Assume first that $m_X \in \{1,2,3,6\}$. Therefore in the decomposition of X into irreducible, faithful K-representations, the subspaces $V_d^{(k)}$ that can appear must have $d \in \{1,2,3,6\}$. By using the tables of contributions given in the proof of Proposition 4.3.12, we see that the eigenvalues of $z^{-1}X$ correspond to singularities of type $\frac{1}{3}(1,1), \frac{1}{6}(1,1), \frac{1}{6}(1,1,1), \frac{1}{6}(1,1,2), \frac{1}{6}(1,1,3), \frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, or $\frac{1}{6}(1,1,1,1,1)$. Indeed, regardless of the choice of value of r we have that the representations leading to a singularity of type $\frac{1}{12}(1,2,7)$ must have either a $V_4^{(0)}$ or $V_{12}^{(k)}$ appear.

For each of these cases, we have one more nontrivial eigenvalue arising from $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$. By Lemma 5.2.2, $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)} \in \{-1, \pm\zeta_3, \pm\zeta_3^2\}$. The smallest contribution of this eigenvalue to $\sum(g)$ is 1/6. Therefore if the eigenvalues arising from $z^{-1}X$ correspond to singularities of type $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,2)$, or $\frac{1}{6}(1,1,1,1,1)$, we have $\sum(g) \geq 1$. For singularities of type $\frac{1}{6}(1,1), \frac{1}{6}(1,1,1), \frac{1}{6}(1,1,2)$, and $\frac{1}{6}(1,1,1,1)$, we can verify that adding one eigenvalue from the set $\{-1, \pm\zeta_3, \pm\zeta_3^2\}$ either forces $\sum(g) \geq 1$ or gives a singularity already listed in the statement of the proposition.

If $z^{-1}X$ is of type $\frac{1}{3}(1,1)$, there is a special case where we can have $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)} = \zeta_6$ so that we would have a singularity of type $\frac{1}{6}(1,2,2)$. But then we would have that g^3 is a quasi-reflection with the nontrivial eigenvalue acting on the coordinate θ. Since we assumed $\langle g \rangle$ does not act with quasi-reflections, we are done with this case.

Now we consider the case where $m_X \in \{4,12\}$. Therefore in the decomposition of X as a K-representation, the subspaces $V_d^{(k)}$ that can appear must have $d \in \{1,2,3,4,6,12\}$. By Proposition 4.3.12, we see that the eigenvalues of $z^{-1}X$ correspond to singularities of type $\frac{1}{3}(1,1), \frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1), \frac{1}{6}(1,1,2), \frac{1}{6}(1,1,3), \frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, or $\frac{1}{12}(1,2,7)$.

As with the previous case, if $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)} = 1$, then we have that all of the singularities on this list can appear at the boundary. We can then assume that $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)} \neq 1$. By Lemma 5.2.3, $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ is a 12th root of unity. Since $m_X \in \{4,12\}$ and that we are still in the case of $r \in \{1,2,3,6\}$ of Proposition 4.3.12, we have that either $z^{-1}X$ contributes eigenvalues corresponding to a singularity of type $\frac{1}{12}(1,2,7)$ or $\sum(g) \geq 1$. Since $\frac{1}{12}(1,2,7)$ already contributes 5/6 to $\sum(g)$, the only possible choice for $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ that does not yield $\sum(g) \geq 1$ is ζ_{12}. This corresponds to a singularity of type $\frac{1}{12}(1,1,2,7)$ at the boundary. $\square$

Remark 5.2.5. The special case mentioned in the proof of Proposition 5.2.4 shows that [Beh11, Corollary 4] may not hold in the case $D = -3$. To be precise, there could be a divisor over a dimension 0 cusp that is pointwise fixed by a nontrivial element of $G(F)$. In other words, there may be branch divisors in the boundary.

We will now consider the elements $g \in G(F)$ such that some power of g acts as a quasi-reflection.

Proposition 5.2.6. *Suppose $\mathbb{B}^n_\Lambda/\Gamma$ is defined by the Eisenstein integers, $n \geq 10$, and $g^k = h \in G(F)$ acts as a quasi-reflection at a boundary point $(0,t) \in \overline{\mathbb{B}^n_\Lambda(F)}$. Then $\overline{\mathbb{B}^n_\Lambda(F)}/G(F)$ has canonical singularities, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g\rangle \subseteq G$ so that $\mathbb{C}^n/\langle g\rangle$ is of type $\frac{1}{3}(1,1), \frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1), \frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3), \frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, $\frac{1}{12}(1,2,7)$, or $\frac{1}{12}(1,1,2,7)$.*

Proof. Suppose that $g^k = h$ acts as a quasi-reflection at the boundary point $(0,t) \in \overline{\mathbb{B}^n_\Lambda(F)}$. As computed previously, the eigenvalues of g acting on the tangent space come from the matrix $z^{-1}X$ and from $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$. We proceed in cases based on where the nontrivial eigenvalue of h comes from.

Suppose first that the nontrivial eigenvalue of h comes from $z^{-1}X$. Since $z^{-1}X$ is an $(n-1)\times(n-1)$ matrix and $n \geq 10$, we can apply Proposition 4.4.11 to conclude that the singularity is either canonical or of one of the usual types. As in the proof of Proposition 5.2.4, we may obtain a contribution of at worst $\frac{1}{12}$ from $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ and therefore we may also obtain a singularity of type $\frac{1}{12}(1,1,2,7)$.

Now suppose that the nontrivial eigenvalue of h comes from $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$. By Lemma 5.2.1 and Lemma 5.2.3, we have that $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ is a twelfth root of unity. By either Proposition 4.3.12 or Proposition 4.4.11, depending on whether some power of $z^{-1}X$ is a quasi-reflection, we have that the contribution from $z^{-1}X$ is either ≥ 1 or yields a singularity from the usual list. By the definition of the modified Reid–Tai sum $\sum'(g)$, the minimal contribution of $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ is $1/6$, as the order of h is either $2, 3, 4$, or 6 and $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ is a twelfth root of unity. These contributions to any of the singularities from our list will either yield canonical singularities or another singularity from the list, except possibly a contribution of $\frac{1}{6}$ of $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ to $\sum'(g)$ with a singularity of type $\frac{1}{3}(1,1)$ from $z^{-1}X$. But from the proof of Proposition 4.3.12 and Proposition 4.4.11 we have that $z^{-1}X$ can only have a singularity of type $\frac{1}{3}(1,1)$ if $m_X \in \{1,2,3,6\}$. Therefore, Lemma 5.2.2 implies that the minimal contribution of $e^{\frac{2\pi i}{\sigma}z^{-1}(vt+w)}$ to $\sum'(g)$ is actually $\frac{1}{3}$, so this case is impossible. □

Theorem 5.2.7. *If $n \geq 10$ and $\mathbb{B}^n_\Lambda$ is defined by the Eisenstein integers, then the singularities of the toroidal compactification $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ are either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^n/G$, where the finite group G admits a cyclic subgroup $\langle g\rangle \subseteq G$ so that $\mathbb{C}^n/\langle g\rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3)$, $\frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2)$, $\frac{1}{6}(1,1,1,1,1)$, $\frac{1}{12}(1,2,7)$, or $\frac{1}{12}(1,1,2,7)$.*

Proof. Theorem 4.4.12, Proposition 5.2.4, and Proposition 5.2.6. □

Chapter 6

Toward Resolving Elliptic Obstructions

In this chapter we consider local criteria for extension of pluricanonical forms over the types of noncanonical singularities that can appear on ball quotients defined by the Eisenstein integers. We also show that locally these vanishing conditions can be captured by special divisors given by lattice elements.

6.1 Local Criterion for Extension over Noncanonical Singularities

Given a Γ-invariant section $\eta \in H^0(\mathbb{B}^n_\Lambda, kK_{\mathbb{B}^n_\Lambda})$ that extends to a regular section $\eta \in H^0(\overline{\mathbb{B}^n_\Lambda/\Gamma}, kK_{\overline{\mathbb{B}^n_\Lambda/\Gamma}})$ over the toroidal compactification, when can we guarantee that η extends to a regular section of a resolution of singularities of the toroidal compactification $\overline{\mathbb{B}^n_\Lambda/\Gamma}$? If the singularities of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ are all canonical, then by definition, every such section will extend to a regular section of a resolution of singularities. Since $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ may not have canonical singularities, we are interested in showing that enough pluri-canonical sections extend to regular sections of resolutions of singularities. The following result asserts that we only need to show that sections η extend to a resolution of singularities over every cyclic quotient.

Proposition 6.1.1 ([Tai82, Proposition 3.1])**.** *Let G act linearly on $\mathbb{C}^N$. If η is a Γ-invariant pluricanonical form on $\mathbb{C}^N$, then η extends to a resolution of singularities of $\mathbb{C}^N/G$ if and only if η extends to a resolution of singularities of $\mathbb{C}^N/\langle g\rangle$ for all $g \in G$.*

Remark 6.1.2. Appendix A provides a survey of the literature and proofs of generalizations of this result.

Theorem 5.2.7 gives a local description of all possible cyclic quotient singularities for toroidal compactifications of ball quotients defined by the Eisenstein integers when $n \geq 9$.

Definition 6.1.3. Suppose a finite cyclic group $\langle g \rangle$ acts linearly on $\mathbb{C}^N$. Let $\{z_1, \ldots, z_N\}$ form an eigenbasis for g acting on $\mathbb{C}^N$. In other words, the action of g with respect to coordinates given by $\{z_1, \ldots, z_N\}$ is diagonal. The hyperplanes $V(z_1)$, …, $V(z_N)$ are the Reid–Tai divisors of the action.

Lemma 6.1.4. *Let*

$$\eta = f(z_1, \ldots, z_n)\ \frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}}{(z_1 \cdots z_n)^k}$$

be a $\langle g \rangle$-invariant pluricanonical form on $\mathbb{C}^n$ where $\mathbb{C}^n/\langle g \rangle$ is of type $\frac{1}{3}(1,1), \frac{1}{6}(1,1), \frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,2)$, $\frac{1}{6}(1,1,3), \frac{1}{6}(1,1,1,1)$, $\frac{1}{6}(1,1,1,2), \frac{1}{6}(1,1,1,1,1)$, $\frac{1}{12}(1,2,7)$, or $\frac{1}{12}(1,2,1,7)$ and $V(z_i)$ are the Reid–Tai divisors of the action. If $\operatorname{ord}_{z_i}(f) \geq k$ and $\operatorname{ord}_{z_2}(f) \geq 5k$, then η extends to a resolution of singularities of $\mathbb{C}^n/\langle g \rangle$.

Remark 6.1.5. The choice of stating the final singularity type as $\frac{1}{12}(1,2,1,7)$ rather than as $\frac{1}{12}(1,1,2,7)$ and taking z_2 as the coordinate with additional order of vanishing will be made evident in Section 6.2, in particular, in Lemma 6.2.7.

Proof. If $\eta = f_0(z_1, \ldots, z_n)\ (dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}$ is a $\langle g \rangle$-invariant pluricanonical form, then since log canonical forms will be easier to work with, define $f(z_1, \ldots, z_n) = (z_1 \cdots z_n)^k f_0(z_1, \ldots, z_n)$ so that we have that

$$\eta = f(z_1, \ldots, z_n)\ \frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}}{(z_1 \cdots z_n)^k}.$$

Note that $\operatorname{ord}_{z_j}(f) = k + \operatorname{ord}_{z_j}(f_0)$. We proceed by following the argument found in the proof of [Daw25, Lemma 6.1]. Essentially, we apply the following three facts:

- We can algorithmically compute a toric resolution of singularities $\pi : Y \to \mathbb{C}^n/\langle g \rangle$.
- The pullback of the log canonical form under π is still the log canonical form.

- We can compute the order of vanishing of f on each affine chart of the blow up using toric geometry.

The statement for toric resolutions of of singularities of cyclic quotient singularities can be obtained by combining the second exercise on page 35 and the proposition on page 48 of [Ful97] or [CLS12, Theorem 11.1.9]. Algorithms for computing these resolutions of singularities are available in the Macaulay2 package NormalToricVarieties. We do this concretely in an example below.

The fact that $\pi^*(d\log z)^k = (d\log w)^k$ is related to the fact that normal toric varieties are log smooth. Concretely, we can verify this on each affine chart of the toric resolution of singularities in each example. We do this in an example below.

The third fact will make itself evident as we compute resolutions of singularities for our specific examples of cyclic quotient singularities.

Let's start by considering a singularity of type $\frac{1}{3}(1,1)$. Let N be the lattice generated by $p_1 = (1,0)$ and $p_2 = (0,1)$. Let $\overline{N} \subset N \otimes \mathbb{Q}$ be the overlattice given by $\overline{N} = N + \frac{1}{3}(1,1)$. Let σ be the standard cone generated by p_1 and p_2, i.e.

$$\sigma = \left\{ \sum_{i=1}^{2} \lambda_i p_i : \lambda_i \geq 0 \right\} \subset \overline{N} \otimes \mathbb{R}.$$

We note that $q_1 = \frac{1}{3}(1,1)$ and $q_2 = p_2$ form a $\mathbb{Z}$-basis for $\overline{N}$; with respect to this basis, σ is generated by $3q_1 - q_2$ and q_2, i.e., $(3,-1)$,$(0,1)$ with respect to the q_1, q_2 basis.

We now resolve the toric variety X_σ by subdividing the cone σ into a fan Σ such that each maximal dimensional face of Σ is smooth, i.e., for every cone of maximal dimension, the minimal lattice points on the extremal rays form a $\mathbb{Z}$-basis for $\overline{N}$. We use the package NormalToricVarieties in Macaulay2 to compute that we can take Σ to be the fan with rays on the basis $\{q_i\}$ given by $u_1 = (3,-1)$,$u_2 = (0,1)$, $u_3 = (1,0)$, having two maximal cones $\{u_1, u_3\}$ and $\{u_2, u_3\}$. Code for this computation can be found in Appendix B.2.

The following figures give a visualization of this computation. Note that all of the following figures show the N lattice in the language of [Ful97], the dual lattice to the lattice of monomials. We begin with the standard lattice and cone for $\mathbb{A}^2$ as a toric variety as seen in Figure 6.1.

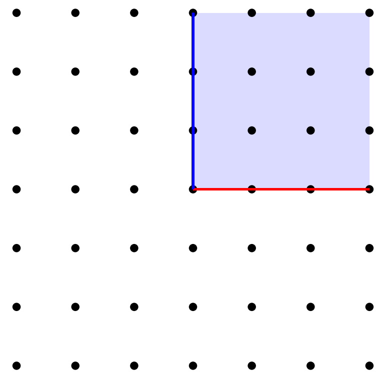

Figure 6.1: Standard lattice and cone for $\mathbb{A}^2$

To construct the quotient, we expand the lattice by adding the point $\frac{1}{3}p_1 + \frac{1}{3}p_2$ and all points generated as seen in Figure 6.2.

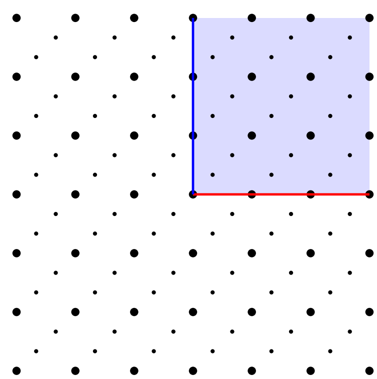

Figure 6.2: Lattice and cone for a $\frac{1}{3}(1,1)$ singularity

In order to construct a resolution of this singularity, we change basis by letting $q_1 = \frac{1}{3}p_1 + \frac{1}{3}p_2$ and $q_2 = p_2$. We compute that $p_1 = 3q_1 - q_2$. Therefore, with respect to the new basis, our toric variety is given by the cone in Figure 6.3.

In order to resolve the singularity, we add a ray in the direction of q_1 so that each subcone of the resulting fan is smooth as in Figure 6.4.

The dual of Figure 6.1 is itself, while the dual of Figure 6.2 is given in Figure 6.5. In other words, Figure 6.5 gives the lattice M and dual cone in the notation of [Ful97]. Therefore the inclusion of rings $\mathbb{C}[x^3, x^2y, xy^2, y^3] \hookrightarrow \mathbb{C}[x,y]$ induces the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$. We let

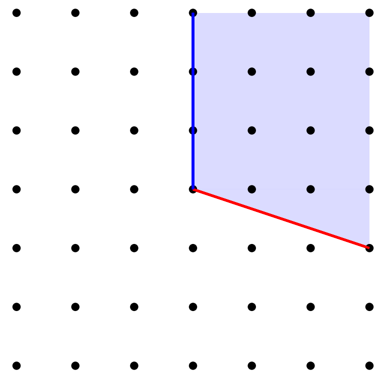

Figure 6.3: Lattice and cone for a $\frac{1}{3}(1,1)$ singularity after changing lattice generators

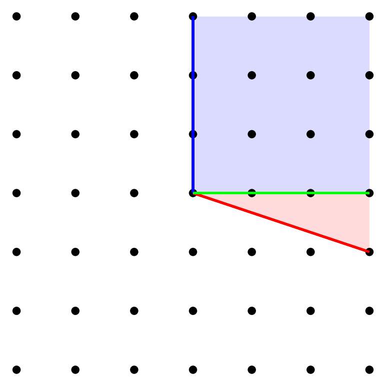

Figure 6.4: Lattice and fan for a resolution of a $\frac{1}{3}(1,1)$ singularity

$X = x^3$, $Y = x^2y$, $Z = xy^2$, and $W = y^3$.

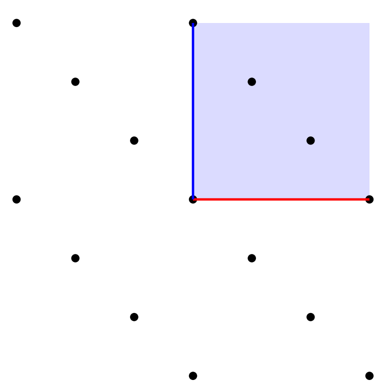

Figure 6.5: Monomial lattice for a $\frac{1}{3}(1,1)$ singularity

Taking duals for Figure 6.4 results in Figure 6.6. Note that the overlapping cones of Figure 6.6 give two affine charts on the resolution of the $\frac{1}{3}(1,1)$ singularity. We let $\mathbb{C}[u,v]$ denote the ring of functions for the affine chart given by the first quadrant of Figure 6.6 and we let $\mathbb{C}[a,b]$ denote the ring of functions for the other affine chart, where, pictorially, u is given by the vertex at $(1,0)$, v is given by the vertex at $(0,1)$, a is given by the vertex at $(1,3)$, and b is given by the vertex at $(0,-1)$. Note that the cone given by the intersection of the two cones giving the affine charts in Figure 6.6

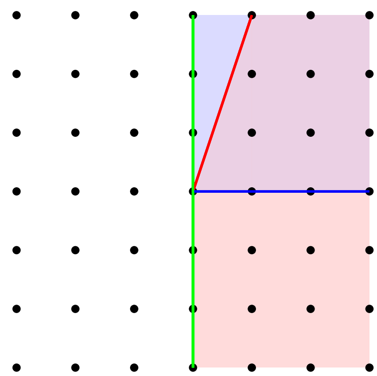

Figure 6.6: Monomial lattice for a $\frac{1}{3}(1,1)$ singularity

is the dual of Figure 6.3. From this we obtain the inclusions of rings $\mathbb{C}[X,Y,Z,W] \hookrightarrow \mathbb{C}[u,v]$ via $X = uv^3$, $Y = uv^2$, $Z = uv$, $W = u$ and $\mathbb{C}[X,Y,Z,W] \hookrightarrow \mathbb{C}[a,b]$ via $X = a$, $Y = ab$, $Z = ab^2$, $W = ab^3$. We summarize this in the following diagram.

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d] & & \mathbb{C}[x,y] & \\
\mathbb{C}^2/G & & \mathbb{C}[X,Y,Z,W] \arrow[u, hook] \arrow[dl, hook] \arrow[dr, hook] & \\
\widetilde{\mathbb{C}^2/G} \arrow[u] & \mathbb{C}[u,v] & & \mathbb{C}[a,b]
\end{tikzcd}$$

We now compute

$$\begin{aligned}
\frac{(du \wedge dv)^{\otimes k}}{(uv)^k} &= \frac{(dW \wedge d(Z/W))^{\otimes k}}{Z^k} \\
&= \frac{(d(y^3) \wedge d(x/y))^{\otimes k}}{(xy^2)^k} \\
&= \frac{(-3y\ dx \wedge dy)^{\otimes k}}{(xy^2)^k}
\end{aligned}$$

$$= (-3)^k \frac{(dx \wedge dy)^{\otimes k}}{(xy)^k}.$$

Note that this is what we mean when we say that the pullback of the log canonical form under the toric resolution of singularities is still the log canonical form. The factor of $(-3)^k$ will have no effect on the order of vanishing of f, we will omit it moving forward. For the other chart, we get a similar computation that we include for completeness.

$$\begin{aligned}
\frac{(da \wedge db)^{\otimes k}}{(ab)^k} &= \frac{(dX \wedge d(Y/X))^{\otimes k}}{Y^k} \\
&= \frac{(d(x^3) \wedge d(y/x))^{\otimes k}}{(x^2y)^k} \\
&= \frac{(3x\ dx \wedge dy)^{\otimes k}}{(x^2y)^k} \\
&= 3^k \frac{(dx \wedge dy)^{\otimes k}}{(xy)^k}.
\end{aligned}$$

We can use these coordinates to see whether a G-invariant pluricanonical form will extend to this resolution of singularities. Let

$$\eta_1 = x^3y^3\ \frac{(dx \wedge dy)^{\otimes 3}}{(xy)^3}.$$

In this case, $f(x, y) = x^3y^3 = XW = u^2v^3 = f(u, v)$. Therefore,

$$\eta_1 = u^2v^3\ \frac{(du \wedge dv)^{\otimes 3}}{(uv)^3}.$$

We see that η_1 does not define a regular pluricanonical form on the resolution of singularities, indeed, η_1 has a pole along $u = 0$.

Note that the coordinate u corresponds to the lattice element $q_1 = \frac{1}{3}p_1 + \frac{1}{3}p_2 \in \overline{N}$ and we have that $\mathrm{ord}_u(f(u, v)) = 2 = \frac{1}{3}\mathrm{ord}_x(f(x, y)) + \frac{1}{3}\mathrm{ord}_y(f(x, y))$.

As a second example, let

$$\eta_2 = x^6y^3\ \frac{(dx \wedge dy)^{\otimes 3}}{(xy)^3}.$$

In this case, $f(x, y) = x^3y^6 = XW^2 = u^3v^3 = f(u, v)$. Therefore,

$$\eta_2 = u^3v^3\ \frac{(du \wedge dv)^{\otimes 3}}{(uv)^3}.$$

On the other chart, $f(x,y) = XW^2 = a^3b^6$, so

$$\eta_2 = a^3b^6 \ \frac{(da \wedge db)^{\otimes 3}}{(ab)^3}.$$

We see that η_2 extends to the resolution of singularities.

See [Rei85, p. 349-353] for additional background and related computations to these examples.

In general, if $z_1^*, \ldots, z_n^*$ are the coordinates of an affine chart of the resolution of a finite cyclic quotient singularity, we obtain similar maps of rings as in the following diagram.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[d] & \mathbb{C}[z_1, \ldots, z_n] \\
\mathbb{C}^n/G & \mathbb{C}[z_1, \ldots, z_n]^G \arrow[u, hook] \arrow[d, hook] \\
\widetilde{\mathbb{C}^n/G} \arrow[u] & \mathbb{C}[z_1^*, \ldots, z_n^*]
\end{tikzcd}$$

Then,

$$\begin{aligned}
\eta &= f(z_1, \ldots, z_n)\frac{(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}}{(z_1 \ldots z_n)^k} \\
&= f(z_1^*, \ldots, z_n^*)\frac{(dz_1^* \wedge \cdots \wedge dz_n^*)^{\otimes k}}{(z_1^* \cdots z_n^*)^k}.
\end{aligned}$$

By $f(z_1^*, \ldots, z_n^*)$, we mean that f is G-invariant, so we consider $f \in \mathbb{C}[z_1, \ldots, z_n]^G \subseteq \mathbb{C}[z_1^*, \ldots, z_n^*]$. Note that η is regular on the resolution of singularities if $\mathrm{ord}_{z_i^*}(f) \geq k$ for all i.

In general, as stated in [Tai82], $\mathrm{ord}_{z_i^*}(f) = \sum_{j=1}^n \lambda_j \mathrm{ord}_{z_j}(f)$ where $v = \sum_{j=1}^n \lambda_j p_j$ is the lattice point of $\overline{N}$ corresponding to the coordinate z_i^*. We showed this in the example with η_1. In general, we obtain this fact by relating the inclusion $f \in \mathbb{C}[z_1, \ldots, z_n]^G \hookrightarrow \mathbb{C}[z_1^*, \ldots, z_n^*]$ to the monomial lattices.

We now prove the result in the case of a $\frac{1}{3}(1,1)$ singularity. Let η satisfy the hypotheses of the lemma, i.e., η is a G-invariant k-form, $\mathrm{ord}_x(f) \geq k$ and $\mathrm{ord}_y(f) \geq 5k$. In order to verify $\pi^*\eta$ is regular on the resolution of singularities X_Σ, we only need to show that $\pi^*\eta$ is regular along the exceptional divisor of X_Σ. Therefore, it suffices to examine $\pi^*\eta$ under the chart $\mathbb{C}[u,v]$. The exceptional divisor is given by $u = 0$, so we compute $\mathrm{ord}_u(f(u,v)) = \frac{1}{3}\mathrm{ord}_x(f(x,y)) + \frac{1}{3}\mathrm{ord}_y(f(x,y)) \geq 2k$.

Therefore,

$$\pi^*\eta = f(u,v)\ \frac{(du \wedge dv)^{\otimes k}}{(uv)^k}$$

is regular on X_Σ.

We remark that adding a smooth component will not change this computation. We proceed in all cases without a smooth component.

We now consider the case $\frac{1}{6}(1,1)$. Let N be the lattice generated by $p_1 = (1,0)$ and $p_2 = (0,1)$. Let $\overline{N} \subset N \otimes \mathbb{Q}$ be the overlattice given by $\overline{N} = N + \frac{1}{6}(1,1)$. Let σ be the standard cone generated by p_1 and p_2.

The elements $q_1 = \frac{1}{6}(1,1)$ and $q_2 = p_2$ form a $\mathbb{Z}$-basis for $\overline{N}$. With respect to these generators, σ is generated by $(6,-1)$ and $(0,1)$.

The fan Σ for a resolution of singularities of X_σ is given by rays on the basis $\{q_i\}$ as $u_1 = (6,-1)$,$u_2 = (0,1)$, and $u_3 = (1,0)$, and maximal cones $\{u_1,u_3\}$ and $\{u_2,u_3\}$. As in the previous case, we only need to consider $\mathrm{ord}_{z_2^*}(f) = \frac{1}{6}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) \geq \frac{k+5k}{6} \geq k$. Therefore, η is regular on X_Σ.

The cases for singularities of type $\frac{1}{6}(1,1,1)$, $\frac{1}{6}(1,1,1,1)$, and $\frac{1}{6}(1,1,1,1,1)$ are all similar to the previous two cases. In fact, the only difference is that there are now 3, 4, and 5 affine charts for the toric resolution of singularities respectively. All of these cases are resolved by adding in the ray $u_{n+1} = (1,0,\ldots,0)$ corresponding to q_1. Therefore we have that $\mathrm{ord}_{z_n^*}(f) \geq 7k/6, 8k/6, 9k/6$ in each respective case.

The cases of singularities of type $\frac{1}{6}(1,1,2)$ and $\frac{1}{6}(1,1,1,2)$ are computed in the proof of [Daw25, Lemma 6.1]. In particular, even if we assumed $\mathrm{ord}_{z_2}(f) \geq 3k$, then η would extend to a resolution of singularities.

We now consider the case $\frac{1}{6}(1,1,3)$. Let N be the lattice generated by $p_1 = (1,0,0)$, $p_2 = (0,1,0)$, and $p_3 = (0,0,1)$. Let $\overline{N} \subset N \otimes \mathbb{Q}$ be the overlattice given by $\overline{N} = N + \frac{1}{6}(1,1,3)$. Let σ be the standard cone generated by p_1, p_2, and p_n.

The elements $q_1 = \frac{1}{6}(1,1,3)$, $q_2 = p_2$, and $q_3 = p_3$ form a set of generators for $\overline{N}$. With respect to these generators, σ is generated by $(6,-1,-3)$,$(0,1,0)$, and $(0,0,1)$.

The fan Σ for a resolution of singularities of X_σ is given by rays on the basis $\{q_i\}$ as $u_1 = (6,-1,-3)$,$u_2 = (0,1,0)$, $u_3 = (0,0,1)$, $u_4 = (1,0,0)$, $u_5 = (2,0,-1)$. There are four maximal cones. We now have two rays corresponding to affine coordinates for exceptional divisors on the resolution of singularities. First, $\mathrm{ord}_{z_1^*}(f) = \frac{1}{6}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) + \frac{1}{2}\mathrm{ord}_{z_3}(f) \geq \frac{3}{2}k$. For the second coordinate, $\mathrm{ord}_{z_n^*}(f) = \frac{2}{3}\mathrm{ord}_{z_1}(f) + \frac{2}{3}\mathrm{ord}_{z_2}(f) \geq 4k$. Therefore, η is regular on X_Σ.

We now consider the case $\frac{1}{12}(1,2,7)$. Let N be the lattice generated by $p_1 = (1,0,0)$, $p_2 = (0,1,0)$, and $p_3 = (0,0,1)$. Let $\overline{N} \subset N \otimes \mathbb{Q}$ be the overlattice given by $\overline{N} = N + \frac{1}{12}(1,2,7)$. Let σ be the standard cone generated by $p_1, \dots, p_n$.

The elements $q_1 = \frac{1}{12}(1,2,7)$, $q_2 = p_2$, ..., $q_n = p_n$ form a $\mathbb{Z}$-basis for $\overline{N}$. With respect to the $\{q_i\}$ basis, σ is generated by $(12,-2,-7)$,$(0,1,0)$, and $(0,0,1)$.

The fan Σ for a resolution of singularities of X_σ is given by rays on the basis $\{q_i\}$ as $u_1 = (12,-2,-7)$,$u_2 = (0,1,0)$, $u_3 = (0,0,1)$, $u_4 = (1,0,0)$, $u_5 = (6,-1,-3)$, $u_6 = (2,0,-1)$, $u_7 = (7,-1,-4)$. There are eight maximal cones. We now have four rays corresponding to affine coordinates for exceptional divisors on the resolution of singularities. For the coordinate corresponding to u_4, $\mathrm{ord}_{z_1^*}(f) = \frac{1}{12}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) + \frac{7}{12}\mathrm{ord}_{z_3}(f) \geq \frac{3}{2}k$. For the coordinate corresponding to u_5, $\mathrm{ord}_{z_n^*}(f) = \frac{1}{2}\mathrm{ord}_{z_1}(f) + \frac{1}{2}\mathrm{ord}_{z_3}(f) \geq k$. For the coordinate corresponding to u_6, $\mathrm{ord}_{z_n^*}(f) = \frac{1}{6}\mathrm{ord}_{z_1}(f) + \frac{1}{3}\mathrm{ord}_{z_2}(f) + \frac{1}{6}\mathrm{ord}_{z_3}(f) \geq 2k$. For the coordinate corresponding to u_7, $\mathrm{ord}_{z_n^*}(f) = \frac{7}{12}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) + \frac{1}{12}\mathrm{ord}_{z_3}(f) \geq \frac{3}{2}k$. Therefore, η is regular on X_Σ.

The case $\frac{1}{12}(1,2,1,7)$ is similar to the previous case. Assuming the usual setup, the fan Σ for a resolution of singularities of X_σ is given by rays on the basis $\{q_i\}$ as $u_1 = (12,-2,-1,-7)$,$u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = (1,0,0,0)$, $u_6 = (6,-1,0,-3)$, $u_7 = (2,0,0,-1)$, $u_8 = (7,-1,0,-4)$. There are fourteen maximal cones. We have four rays corresponding to affine coordinates for exceptional divisors on the resolution of singularities. For the coordinate corresponding to u_5, $\mathrm{ord}_{z_1^*}(f) = \frac{1}{12}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) + \frac{1}{12}\mathrm{ord}_{z_3}(f) + \frac{7}{12}\mathrm{ord}_{z_4}(f) \geq \frac{19}{12}k$. For the coordinate corresponding to u_6, $\mathrm{ord}_{z_n^*}(f) = \frac{1}{2}\mathrm{ord}_{z_1}(f) + \frac{1}{2}\mathrm{ord}_{z_3}(f) \geq k$. For the coordinate corresponding to u_7, $\mathrm{ord}_{z_n^*}(f) = \frac{1}{6}\mathrm{ord}_{z_1}(f) + \frac{1}{3}\mathrm{ord}_{z_2}(f) + \frac{1}{6}\mathrm{ord}_{z_3}(f) + \frac{1}{6}\mathrm{ord}_{z_4}(f) \geq \frac{13}{6}k$. For the coordinate corresponding to u_8, $\mathrm{ord}_{z_n^*}(f) = \frac{7}{12}\mathrm{ord}_{z_1}(f) + \frac{1}{6}\mathrm{ord}_{z_2}(f) + \frac{7}{12}\mathrm{ord}_{z_3}(f) + \frac{1}{12}\mathrm{ord}_{z_3}(f) \geq \frac{25}{12}k$. Therefore, η is regular on X_Σ. □

6.2 Special Divisors

In this section, we show that the coordinates we require vanishing along for extension of pluricanonical forms over noncanonical singularities can be described via lattice elements. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}$ and let Λ be a Hermitian $\mathcal{O}$-lattice of signature $(1, n)$. Let

$$\mathbb{B}_\Lambda := \{[x] \in \mathbb{P}(\Lambda \otimes_{\mathcal{O}} \mathbb{C}) \mid h(x, x) > 0\}$$

denote the ball.

Definition 6.2.1. Let $v \in \Lambda$ with $v^2 < 0$, then

$$\mathbb{B}_\Lambda(v) = \{[x] \in \mathbb{B}_\Lambda \mid h(x, v) = 0\}.$$

We call such sets $\mathcal{O}$-hyperplane divisors of $\mathbb{B}_\Lambda$.

Lemma 6.2.2. *Every $\mathcal{O}$-hyperplane divisor is the intersection of a hyperplane in $\mathbb{P}(\Lambda \otimes_{\mathcal{O}} \mathbb{C})$ with $\mathbb{B}^n_\Lambda$.*

Proof. Fix a basis for Λ. This extends to a $\mathbb{C}$-basis for $\Lambda_{\mathbb{C}}$. Let H be the matrix for the Hermitian form $\langle , \rangle$ with respect to this basis. Since H is defined over $\mathcal{O}$, we know that $H \in M_{n+1}(K)$. Now we have

$$\begin{aligned}
\mathbb{B}_\Lambda(v) &= \{[x] \in \mathbb{B}^n_\Lambda \mid h(x, v) = 0\} \\
&= \left\{ [x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \begin{pmatrix} x_0 & \cdots & x_n \end{pmatrix} H \begin{pmatrix} v_0 \\ \vdots \\ v_n \end{pmatrix} = 0 \right\} \cap \mathbb{B}^n_\Lambda \\
&= \left\{ [x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \begin{pmatrix} x_0 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \widetilde{v_0} \\ \vdots \\ \widetilde{v_n} \end{pmatrix} = 0 \right\} \cap \mathbb{B}^n_\Lambda
\end{aligned}$$

$$= V(\widetilde{v_0}x_0 + \cdots + \widetilde{v_n}x_n) \cap \mathbb{B}^n_\Lambda,$$

where $H \in M_{n+1}(K)$, $v_i \in \mathcal{O} \Rightarrow \widetilde{v_i} \in K$. By multiplying by a common denominator, we can force the $\widetilde{v_i}$ to be elements of $\mathcal{O}$. $\square$

Let Γ be a finite index subgroup of $U(\Lambda)$.

Definition 6.2.3. We call an element $g \in \Gamma$ **elliptic at $[x] \in \mathbb{B}^n_\Lambda$** if $\mathbb{B}^n_\Lambda/\langle g\rangle$ has a noncanonical singularity at $\overline{[x]}$. In other words, g is elliptic at $[x]$ if and only if $g \in \mathrm{Stab}_\Gamma([x])$ and the appropriate Reid–Tai sum is < 1, i.e. if no power of g is a quasi-reflection, $\sum(g) < 1$, otherwise $\sum'(g) < 1$.

Recall that $\mathbb{B}^n_\Lambda$ is a $\mathbb{C}$-manifold and so we can consider $\mathbb{C}$-analytic divisors on $\mathbb{B}^n_\Lambda$.

Definition 6.2.4. We call a $\mathbb{C}$-analytic divisor $D \subset \mathbb{B}^n_\Lambda$ a **Reid–Tai (RT) divisor for $g \in \Gamma$** if g is elliptic for some $[x] \in D$ and, in local analytic coordinates, $D = V(x_i)$ where $\frac{\partial}{\partial x_i} \in T_{[x]}\mathbb{B}^n_\Lambda$ is part of an eigenbasis that diagonalizes the action of g on $T_{[x]}\mathbb{B}^n_\Lambda$ and g does not fix $\frac{\partial}{\partial x_i}$.

Example 6.2.5. Let $\mathcal{O} = \mathbb{Z}[\zeta_3]$ and $\Lambda = \langle 1\rangle \oplus \langle -1\rangle^{\oplus 2}$ be the $\mathcal{O}$-lattice with the standard Hermitian form of signature $(1,2)$. Let $v = (0,1,0) \in \Lambda$. Then

$$\begin{aligned}
\mathbb{B}_\Lambda(v) &= \{[x] \in \mathbb{B}^2_\Lambda \mid \langle x, v\rangle = 0\} \\
&= \{[1 : x_0 : x_1] \in \mathbb{B}^2_\Lambda \mid \langle (1, x_0, x_1), (0,1,0)\rangle = 0\} \\
&= \{[1 : x_0 : x_1] \in \mathbb{B}^2_\Lambda \mid -x_0 = 0\} \\
&= \{[1 : 0 : x_1] \in \mathbb{B}^2_\Lambda\}
\end{aligned}$$

Note that the final line shows that $|x_1| < 1$ and this clearly has codimension 1 in $\mathbb{B}_\Lambda$. Since (x_0, x_1) give local analytic coordinates on $\mathbb{B}^2_\Lambda$, this is the $\mathbb{C}$-analytic divisor $V(x_0)$.

Example 6.2.6. Now consider $\Gamma = U(\Lambda)$ and let

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix},$$

with respect to the standard basis. Then g stabilizes $[1:0:0] \in \mathbb{B}^2_\Lambda$. The tangent space at $[1:0:0]$ is given by $\mathrm{Hom}(\mathbb{C}\cdot(1,0,0), \Lambda_\mathbb{C}/\mathbb{C}\cdot(1,0,0)) \cong \mathbb{C}^2$ and g acts by sending $(x,y) \mapsto (\zeta_3 x, \zeta_3 y)$, where x is the homomorphism sending $(1,0,0)$ to $(0,1,0)\mathbb{C}\cdot(1,0,0)$ and y is the homomorphism sending $(1,0,0)$ to $(0,0,1)+\mathbb{C}\cdot(1,0,0)$. By thinking of x and y as the local coordinates they determine, $V(x)$ and $V(y)$ are RT divisors associated to g and $[1:0:0]$. In this example we see that $V(x) = \mathbb{B}_\Lambda((0,1,0))$.

Note that there are many RT divisors associated to g and $[1:0:0]$ in this example. Indeed $V(ax+by)$ is a RT divisor for each $a, b \in \mathbb{C}$. Most of these RT divisors cannot be described as an $\mathcal{O}$-hyperplane divisor.

The next result shows that for elliptic elements we can always choose a RT divisor that is also an $\mathcal{O}$-hyperplane divisor.

Lemma 6.2.7. *Suppose $g \in \Gamma$ is elliptic at $[x] \in \mathbb{B}^n_\Lambda$ and $n \geq 9$. Then there exists a RT divisor D containing $[x]$ such that D is an $\mathcal{O}$-hyperplane divisor.*

Proof. If $g \in \Gamma$ is elliptic at $[x] \in \mathbb{B}^n_\Lambda$, then $\overline{[x]} \in \mathbb{B}^n_\Lambda/\langle g\rangle$ is a finite quotient singularity of one of the types listed in Theorem 4.4.12. Let λ be the eigenvalue of g acting on $\mathbb{W} = \mathrm{span}_\mathbb{C}(x) \subseteq \Lambda_\mathbb{C}$. By Corollary 4.4.13, in all cases there is some $V_d^{(k)}$ appearing in the decomposition such that $\dim V_d^{(k)} = 1$ and the eigenvalue for $V_d^{(k)}$ is not λ.

Since $\dim V_d^{(k)} = 1$, let $v \in \Lambda$ be a minimal generator of $V_d^{(k)}$. Since $[x] \in \mathbb{B}^n_\Lambda$, we have that $h(x,x) > 0$. Since g acts nontrivially on the tangent space at $[x]$ in the direction of v, we have that $g\cdot x = \lambda x$, $g \cdot v = \lambda' v$, and $\lambda \neq \lambda'$. Moreover, by the proofs of Proposition 4.3.12 and Proposition 4.4.11, $\lambda, \lambda' \in \mathcal{O}^\times$, so $\overline{\lambda_j} = \lambda_j^{-1}$. Therefore,

$$h(x,v) = h(\lambda_1 x, \lambda_2 v) = \lambda_1\lambda_2^{-1}h(x,v).$$

Thus $h(x,v) = 0$ since $\lambda_1 \neq \lambda_2$. Since v is orthogonal to x, $h(x,x) > 0$, h has signature $(1,n)$, $h(v,v) < 0$.

We define the $\mathcal{O}$-hyperplane divisor $\mathbb{B}_\Lambda(v)$. By Lemma 6.2.2, $\mathbb{B}_\Lambda(v) = V(H) \cap \mathbb{B}^n_\Lambda$, where

$H \in \mathcal{O}[Z_0, \dots, Z_n]_1$, degree 1 homogeneous polynomials with coefficients in $\mathcal{O}$. By taking a suitable affine chart on $\mathbb{P}\Lambda_{\mathbb{C}}$, H determines an affine coordinate z such that, locally, $V(H) = V(z) = \mathbb{B}_\Lambda(v)$.

By the canonical isomorphism

$$T_{[x]}\mathbb{P}\Lambda_{\mathbb{C}} \cong \mathrm{Hom}(\mathrm{span}_{\mathbb{C}}\{x\}, \Lambda_{\mathbb{C}}/\mathrm{span}_{\mathbb{C}}\{x\}),$$

we have that $\frac{\partial}{\partial z}$ is an eigenvector of the action of g that can be extended to a $\mathbb{C}$-basis diagonalizing the action of g. $\square$

Proposition 6.2.8. *Suppose $\eta \in H^0(\mathbb{B}^n_\Lambda, kK_{\mathbb{B}^n_\Lambda})$ defines a form $\eta \in H^0(\overline{\mathbb{B}^n_\Lambda/\Gamma}, kK_{\overline{\mathbb{B}^n_\Lambda/\Gamma}})$, where $\mathbb{B}^n_\Lambda$ and Γ are defined by the Eisenstein integers and $n \geq 9$. Let $\overline{[\omega]} \in \mathbb{B}^n_\Lambda/\Gamma$ be a noncanonical singularity on the interior of the ball quotient. There exists a finite set $\mathcal{D}$ of $\mathcal{O}$-hyperplane divisors such that if η vanishes to order at least $6k$ along each $D \in \mathcal{D}$, then η extends to a regular pluricanonical form on a resolution of singularities of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$ near $\overline{[\omega]}$.*

Remark 6.2.9. Not every Γ-invariant form $\eta \in H^0(\mathbb{B}^n_\Lambda, kK_{\mathbb{B}^n_\Lambda})$ descends to a regular form on $\mathbb{B}^n_\Lambda/\Gamma$. Moreover, not every such form extends to the toroidal boundary of $\overline{\mathbb{B}^n_\Lambda/\Gamma}$. These kinds of questions are addressed in [Mae23] and [Mae24]. The results of this thesis may provide a means of loosening the restriction of (5) in [Mae23, Theorem 6.3] to show that many ball quotients defined by the Eisenstein integers, which may not have canonical singularities, are of general type.

Proof. Let G denote the stabilizer of $[x]$ in $\mathbb{B}^n_\Lambda$. By Lemma 2.1.2, $|G| < \infty$. By [Tai82, Proposition 3.1], η locally extends to a resolution of singularities if and only if η extends to a resolution of singularities for $\mathbb{B}^n_\Lambda/\langle g\rangle$ for each $g \in G$. So it suffices to show that η extends to a resolution of singularities over each cyclic subquotient. If $\mathbb{B}^n_\Lambda/\langle g\rangle$ has canonical singularities, then η extends to a resolution of singularities. By Theorem 5.2.7, we know all possible noncanonical singularities. By Lemma 6.2.7, each $g \in G$ leading to a noncanonical singularity corresponds to an $\mathcal{O}$-hyperplane divisor D_g that is also a RT divisor. Since $|G| < \infty$ there are finitely many such divisors. By Lemma 6.1.4, the vanishing of η to order at least $6k$ along each D_g guarantees that η extends to a resolution of singularities for $\mathbb{B}^n_\Lambda/\langle g\rangle$ for each $g \in G$. $\square$

Remark 6.2.10. For $x \in \overline{\mathbb{B}^n_\Lambda/\Gamma}$ in the toroidal boundary, we suspect a similar statement can be made by extending $\mathcal{O}$-hyperplane divisors to the boundary. It is currently unclear how to relate these divisors to the eigenspaces given by $z^{-1}X$ in the proof of Proposition 5.2.4 and Proposition 5.2.6.

Chapter 7

Generalized Ball Quotients

In this section we consider singularities on locally symmetric domains arising from Hermitian symmetric domains of type $I_{m,n}$ with $m \leq n$. Note that ball quotients are of type $I_{1,n}$.

7.1 Generalized Ball Quotients of Type $(2, n)$

We first consider the generalized ball of type $(2, n)$. To construct such an example, let $\mathcal{O}$ be the ring of integers of an imaginary quadratic number field K and $\Lambda \cong \mathcal{O}^{\oplus n+2}$ be a lattice with Hermitian form $h : \Lambda \times \Lambda \to \mathcal{O}$ of signature $(2, n)$. We define the generalized ball $\mathbb{B}_\Lambda(2, n)$ as the 2-dimensional subspaces W of $\Lambda_\mathbb{C} = \Lambda \otimes_\mathcal{O} \mathbb{C}$ such that h restricted to W is positive definite.

$$\mathbb{B}_\Lambda(2, n) = \{W \in \mathrm{Gr}(2, \Lambda_\mathbb{C}) : h|_W \text{ is positive definite}\}.$$

We have that $U(\Lambda, h)$ acts on $\mathbb{B}_\Lambda(2, n)$ and that any group $\Gamma < U(\Lambda, h)$ of finite index defines an arithmetic group acting on $\mathbb{B}_\Lambda(2, n)$. Note that the condition of having $h|_W$ be positive definite is an open condition on $\mathrm{Gr}(2, \Lambda_\mathbb{C})$, therefore by the Euler sequence we have a canonical isomorphism:

$$T_{[W]}\mathbb{B}_\Lambda(2, n) \cong \mathrm{Hom}(W, \Lambda_\mathbb{C}/W).$$

For the following discussion, fix some subspace $W \in \mathbb{B}_\Lambda(2, n)$, let $G = \mathrm{Stab}_\Gamma(W)$, and fix some $g \in G$. The first observation we can make is that, just as in the case of $\mathbb{B}_\Lambda(2, n)$, if we know the eigenvalues of the action of g on $\Lambda_\mathbb{C}$ (and so also on W), then we can know the eigenvalues of the action of g on $T_{[W]}\mathbb{B}_\Lambda(2, n)$.

Lemma 7.1.1. *If the eigenvalues of g acting on W are given by λ_1, λ_2 and the eigenvalues of g acting on $\Lambda_{\mathbb{C}}$ are given by $\lambda_1, \lambda_2, \mu_1, \dots, \mu_n$, then the eigenvalues of g acting on $T_{[W]}\mathbb{B}_\Lambda(2, n)$ are given by $\lambda_1^{-1}\mu_1, \lambda_2^{-1}\mu_1 \dots, \lambda_1^{-1}\mu_n, \lambda_2^{-1}\mu_n$.*

Proof. One can appeal to the isomorphism given by the Euler sequence for the tangent space. Instead, we check an affine chart on the Grassmannian. Let $\mathcal{B} = \{w_1, w_2, b_1 \dots, b_n\}$ be a g-eigenbasis for $\Lambda_{\mathbb{C}}$ such that $\{w_1, w_2\}$ forms a basis for W. So we obtain the following frame representing W with respect to $\mathcal{B}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

We can then denote affine coordinates on the Grassmannian in a neighborhood of W as $z_{11}, z_{12}, \dots, z_{n2}$ given as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{11} & z_{12} \\ \vdots & \vdots \\ z_{n1} & z_{n2} \end{pmatrix}.$$

Since B is a g-eigenbasis, we have that

$$g \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{11} & z_{12} \\ \vdots & \vdots \\ z_{n1} & z_{n2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \mu_1 z_{11} & \mu_1 z_{12} \\ \vdots & \vdots \\ \mu_n z_{n1} & \mu_n z_{n2} \end{pmatrix}.$$

Recall that to return to our chosen affine coordinates in a neighborhood of W, we right multiply this matrix by $\mathrm{diag}(\lambda_1, \lambda_2)^{-1}$. Therefore,

$$g \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ z_{11} & z_{12} \\ \vdots & \vdots \\ z_{n1} & z_{n2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \mu_1\lambda_1^{-1}z_{11} & \mu_1\lambda_2^{-1}z_{12} \\ \vdots & \vdots \\ \mu_n\lambda_1^{-1}z_{n1} & \mu_n\lambda_2^{-1}z_{n2} \end{pmatrix}.$$

□

As in the statement of Lemma 7.1.1, let λ_1 and λ_2 be the two (possibly equal) eigenvalues of g acting on W. Let $r_1 = |\lambda_1|$ and $r_2 = |\lambda_2|$. By the results of Chapter 3, we have that

$$\Lambda_K \cong \bigoplus_j V_{d_j}^{(k_j)}.$$

Since λ_1 and λ_2 appear as eigenvalues on Λ_K, we have that either $W \subset V_r^{(k)} \otimes_K \mathbb{C}$, in which case $r_1 = r_2 = r$, or there exists $V_{r_1}^{(k_1)}$ and $V_{r_2}^{(k_2)}$ such that λ_1 appears as an eigenvalue on $V_{r_1}^{(k_1)}$ and λ_2 appears as an eigenvalue on $V_{r_2}^{(k_2)}$. Note that it is not necessarily true that $W \subseteq V_{r_1}^{(k_1)} \otimes_K \mathbb{C} \oplus V_{r_2}^{(k_2)} \otimes_K \mathbb{C}$ because there may be other subspaces of $\Lambda_{\mathbb{C}}$ contributing to the λ_1 and λ_2 eigenspaces.

7.2 Defined by the Eisenstein Integers

For this section, fix $K = \mathbb{Q}(\zeta_3)$. We now want to understand what values of r, r_1, r_2 can appear in the decomposition of Λ_K. We start be considering the case where W is not contained in one $V_r^{(k)}$.

Lemma 7.2.1. *Let $W \in \mathbb{B}_\Lambda(2,n)$ and $g \in \mathrm{Stab}_\Gamma(W)$. Suppose W is not contained in the complexification of one irreducible component of the decomposition of Λ_K as a g-module. On $\mathbb{B}_\Lambda(2,n)$ defined by $\mathbb{Q}(\zeta_3)$, the Reid–Tai sum $\sum(g) \geq 1$ if $r_1 \notin \{1,2,3,4,6,12\}$ or $r_2 \notin \{1,2,3,4,6,12\}$.*

Remark 7.2.2. We obtain a sharper conclusion in Lemma 7.2.3.

Proof. Let λ_1 and λ_2 be the eigenvalues of g acting on W. By assumption, we are in the case where there exists $V_{r_1}^{(k_1)}$ and $V_{r_2}^{(k_2)}$ such that λ_1 appears as an eigenvalue on $V_{r_1}^{(k_1)}$ and λ_2 appears as an eigenvalue on $V_{r_2}^{(k_2)}$. Let $\mu_1, \ldots, \mu_s$ denote the eigenvalues distinct from λ_1 on $V_{r_1}^{(k_1)}$. By Lemma 7.1.1, $\lambda_1^{-1}\mu_1, \ldots, \lambda_1^{-1}\mu_2$ forms a subset of the eigenvalues of g on the tangent space. But these are precisely the eigenvalues considered when proving Lemma 4.3.1. Therefore, by Lemma 4.3.1 we have that $\sum(g) \geq 1$ if $r_1 \notin \{1, 2, 3, 4, 6, 12\}$. The same argument applies to r_2 thus proving the desired result. $\square$

We proceed by continuing to assume that W is not contained in some $V_r^{(k)}$. So the eigenvalues λ_1 and λ_2 of g on W lead to components $V_{r_1}^{(k_1)}$ and $V_{r_2}^{(k_2)}$ appearing in the decomposition of Λ_K as a g-module. Without loss of generality, we assume $r_1 \leq r_2$. In the next lemma, we find the contributions of $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ to $\sum(g)$, the Reid–Tai sum of g acting on $T_W\mathbb{B}_\Lambda(2, n)$.

Lemma 7.2.3. *If $K = \mathbb{Q}(\zeta_3)$, $r_1 \leq r_2$, and $(r_1, r_2) \notin \{1, 2, 3, 6\}^2 \cup \{(1, 4), (1, 12), (2, 4), (2, 12), (3, 4), (3, 12), (4, 6), (6, 12)\}$, then $\sum(g) \geq 1$ on $\mathbb{B}_\Lambda(2, n)$. If $(r_1, r_2) \in \{(1, 2, 3, 6)\}^2$ then the contributions of $V_{r_1}^{(k)}$ and $V_{r_2}^{(j)}$ to the Reid–Tai sum is 0. The contributions of all other choices of eigenvalues contributing less than 1 to the Reid–Tai sum are given in the following table:*

Representation	*Eigenvalues on W*	*Contribution*
$V_1^{(0)} \oplus V_4^{(0)}$	$(1, -i)$	$3/4$
$V_1^{(0)} \oplus V_{12}^{(1)}$	$(1, \zeta_{12}^7)$	$7/12$
$V_1^{(0)} \oplus V_{12}^{(2)}$	$(1, \zeta_{12}^{11})$	$11/12$
$V_2^{(0)} \oplus V_4^{(0)}$	$(-1, i)$	$3/4$
$V_2^{(0)} \oplus V_{12}^{(1)}$	$(-1, \zeta_{12})$	$7/12$
$V_2^{(0)} \oplus V_{12}^{(2)}$	$(-1, \zeta_{12}^5)$	$11/12$
$V_3^{(1)} \oplus V_4^{(0)}$	(ζ_3, i)	$11/12$
$V_3^{(2)} \oplus V_4^{(0)}$	(ζ_3^2, i)	$7/12$
$V_3^{(1)} \oplus V_{12}^{(1)}$	(ζ_3, ζ_{12})	$3/4$
$V_3^{(1)} \oplus V_{12}^{(2)}$	$(\zeta_3, \zeta_{12}^{11})$	$7/12$
$V_3^{(2)} \oplus V_{12}^{(2)}$	$(\zeta_3^2, \zeta_{12}^5)$	$3/4$

$V_3^{(2)} \oplus V_{12}^{(1)}$	$(\zeta_3^2, \zeta_{12}^7)$	11/12
$V_4^{(0)} \oplus V_6^{(1)}$	$(-i, \zeta_6)$	7/12
$V_4^{(0)} \oplus V_6^{(2)}$	$(-i, \zeta_6^5)$	11/12
$V_6^{(1)} \oplus V_{12}^{(1)}$	(ζ_6, ζ_{12})	11/12
$V_6^{(1)} \oplus V_{12}^{(2)}$	$(\zeta_6, \zeta_{12}^{11})$	3/4
$V_6^{(2)} \oplus V_{12}^{(2)}$	$(\zeta_6^5, \zeta_{12}^5)$	7/12
$V_6^{(2)} \oplus V_{12}^{(1)}$	$(\zeta_6^5, \zeta_{12}^7)$	3/4

Proof. We proceed in cases as given before the statement of the lemma. We show the first five cases by hand and appeal to computer calculation for the remaining cases.

(1) $(1, 4)$, we have that the eigenvalues on W are either $(1, i)$ or $(1, -i)$. In the case of $(1, i)$, the other eigenvalue on $V_4^{(0)}$ is $-i$. On the tangent space to the Grassmannian, we then know of two eigenvalues: $1^{-1} \cdot (-i) = -i$ and $i^{-1} \cdot (-i) = -1$. Therefore we have contributed 5/4 to the Reid–Tai sum. Similarly, if the eigenvalues on W are $(1, -i)$, then the eigenvalues on the tangent space are i and $(-i)^{-1}i = -1$, so we contribute 3/4.

(2) $(1, 12)$, we have two choices of representation, either $V_1^{(0)} \oplus V_{12}^{(1)}$ or $V_1^{(0)} \oplus V_{12}^{(2)}$. In each case we have 2 choices of eigenvalues for W.

$V_1^{(0)} \oplus V_{12}^{(1)}$: We can choose eigenvalues $(1, \zeta_{12})$ or $(1, \zeta_{12}^7)$. In the first case, we have that ζ_{12}^{11} and ζ_{12}^7 are both eigenvalues acting on the tangent space, so we have a large enough $\sum(g)$. In the second case, the eigenvalues are ζ_{12} and ζ_{12}^6, so we have a contribution of 7/12.

$V_1^{(0)} \oplus V_{12}^{(2)}$: Similar in this case, we either have $(1, \zeta_{12}^5)$ or $(1, \zeta_{12}^{11})$. In the first case we have a contribution of 18/12 and in the second case we have a contribution of 11/12.

(3) $(2, 4)$, similar to the case of $(1, 4)$. Eigenvalues are $(-1, i)$ or $(-1, -i)$ on W. Therefore on the tangent space we get eigenvalues $i^{-1} \cdot (-i) = -1$ and $-1 \cdot (-i) = i$, so a contribution of 3/4.

In the latter case, we get a contribution of 5/4.

(4) $(2,12)$, similar to the case of $(1,12)$.

$V_2^{(0)} \oplus V_{12}^{(1)}$: Eigenvalues are $(-1,\zeta_{12})$ or $(-1,\zeta_{12}^7)$. The first case has eigenvalues $-1\cdot\zeta_{12}^7 = \zeta_{12}$ and $\zeta_{12}^{11}\cdot\zeta_{12}^7 = \zeta_{12}^6$, so we get a contribution of $7/12$. In the latter the eigenvalues on the tangent space are $-1\cdot\zeta_{12}$ and $\zeta_{12}^5\cdot\zeta_{12}$, so we get a contribution of $13/12$.

$V_2^{(0)} \oplus V_{12}^{(2)}$: Eigenvalues are $(-1,\zeta_{12}^5)$ or $(-1,\zeta_{12}^{11})$. The first case has eigenvalues $-1\cdot\zeta_{12}^{11} = \zeta_{12}^5$ and $\zeta_{12}^7\cdot\zeta_{12}^{11} = \zeta_{12}^6$, so we get a contribution of $11/12$. In the latter the eigenvalues on the tangent space are $-1\cdot\zeta_{12}^5$ and $\zeta_{12}\cdot\zeta_{12}^5$, so we get a contribution of $17/12$.

(5) $(3,4)$:

$V_3^{(1)}\oplus V_4^{(0)}$: Eigenvalues on W can be (ζ_3, i) or $(\zeta_3,-i)$. On the tangent space we get eigenvalues $\zeta_3^2(-i) = \zeta_{12}^8\zeta_{12}^9 = \zeta_{12}^5$ and -1 yielding a contribution of $11/12$. In the second case, we have $\zeta_3^2(i) = \zeta_{12}^{11}$ and -1 yielding a contribution of $17/12$.

$V_3^{(2)} \oplus V_4^{(0)}$: Eigenvalues on W can be (ζ_3^2, i) or $(\zeta_3^2,-i)$. On the tangent space we get eigenvalues $\zeta_3(-i) = \zeta_{12}^4\zeta_{12}^9 = \zeta_{12}$ and -1 yielding a contribution of $7/12$. In the second case, we have $\zeta_3(i) = \zeta_{12}^7$ and -1 yielding a contribution of $13/12$.

For the remaining cases, we use computer calculation. $\square$

Now we investigate the Reid–Tai sum directly for the remaining cases.

Lemma 7.2.4. *Let $W \in \mathbb{B}_\Lambda(2,n)$ and $g\in \mathrm{Stab}_\Gamma(W)$. Suppose W is not contained in the complexification of one irreducible component of the decomposition of Λ_K as a g-module. If $n\geq 6$, then the singularity of $\mathbb{B}_\Lambda(2,n)/\Gamma$ at W is either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^{2n}/G$, where the finite group G admits a cyclic subgroup $\langle g\rangle \subseteq G$ so that $\mathbb{C}^{2n}/\langle g\rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, or $\frac{1}{6}(1,1,1,1)$. When $n=2$, there may also be singularities of type $\frac{1}{6}(1,1,2)$. When $n=3$, there may also be singularities of type $\frac{1}{6}(1,1,1)$ or $\frac{1}{6}(1,1,1,2)$. When $n=5$, there may also be singularities of type $\frac{1}{6}(1,1,1,1,1)$.*

Proof. We consider the contributions of each $V_d^{(j)}$ for each remaining (r_1, r_2) and choice of eigenvalues. By Lemma 4.3.4, we obtain immediately that the possible values of d must be in $\{1, 2, 3, 4, 6, 12\}$ in the case of $K = \mathbb{Q}(\zeta_3)$, otherwise $\sum(g) \geq 1$.

d	contribution
1	0
2	1
3	$2/3, \;\; 4/3$
4	2
6	$1/3, \;\; 5/3$
12	$4/3, \;\; 8/3$

Table 7.2: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 1)$ and $(\lambda_1, \lambda_2) = (1, 1)$

In the case $(r_1, r_2) = (1, 1)$, we have that the eigenvalues of W do not force any contribution and for each $V_3^{(1)}$ and $V_6^{(1)}$ that appear we have a contribution of $2/3$ and $1/3$ respectively. Moreover, we can have as many trivial representations $V_1^{(0)}$ as we wish with no contribution. Ignoring the trivial components, the following representations appearing in the decomposition of Λ_K would yield noncanonical singularities:

(1) $V_3^{(1)} : \frac{1}{3}(1, 1)$

(2) $V_6^{(1)} : \frac{1}{6}(1, 1)$

(3) $V_6^{(1)} \oplus V_6^{(1)}$: $\frac{1}{6}(1, 1, 1, 1)$

d	contribution
1	1/2
2	1/2
3	7/6, 5/6
4	2
6	5/6, 7/6
12	4/3, 8/3

Table 7.3: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 2)$ and $(\lambda_1, \lambda_2) = (1, -1)$

In the case of $(r_1, r_2) = (1, 2)$, we have that the contributions from all d are at least 1/2. By dimension restrictions, for any ball quotient of type $(2, n)$ we must have at least 2 of these d components appearing in the decomposition. Indeed, Λ must be at least rank 4. Therefore, every singularity appearing is canonical.

d	contribution
1	2/3
2	2/3
3	1/3, 1
4	7/3
6	1, 4/3
12	5/3, 2

Table 7.4: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 3)$ and $(\lambda_1, \lambda_2) = (1, \zeta_3)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_1^{(0)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$. Note that this contributes a singularity of type $\frac{1}{3}(1, 1)$ and can only occur when $n = 2$.

d	contribution
1	1/3
2	4/3
3	1, 2/3
4	5/3
6	2/3, 1
12	2, 7/3

Table 7.5: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 3)$ and $(\lambda_1, \lambda_2) = (1, \zeta_3^2)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_1^{(0)} \oplus V_3^{(2)} \oplus V_1^{(0)} \oplus V_1^{(0)}$. Note that this contributes a singularity of type $\frac{1}{3}(1,1)$ and can only occur when $n = 2$.

d	contribution
1	1/4
2	5/4
3	11/12, 19/12
4	3/2
6	7/12, 11/12
12	11/6, 13/6

Table 7.6: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 4)$ and $(\lambda_1, \lambda_2) = (1, -i)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 3/4 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/4, every singularity appearing is canonical.

d	contribution
1	$5/6$
2	$5/6$
3	$1/2,\ \ 7/6$
4	$7/6$
6	$1/6,\ \ 3/2$
12	$2,\ \ 7/3$

Table 7.7: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 6)$ and $(\lambda_1, \lambda_2) = (1, \zeta_6)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. There are six cases of decompositions of Λ_K leading to noncanonical singularities:

(1) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^1 \oplus V_6^{(1)} : \frac{1}{6}(1, 1)$

(2) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^1 \oplus V_6^{(1)} \oplus V_6^{(1)} : \frac{1}{6}(1, 1, 1)$

(3) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^1 \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} : \frac{1}{6}(1, 1, 1, 1)$

(4) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^1 \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} : \frac{1}{6}(1, 1, 1, 1, 1)$

(5) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_3^{(1)} : \frac{1}{6}(1, 1, 2)$

(6) $V_1^{(0)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_3^{(1)} : \frac{1}{6}(1, 1, 1, 2)$

d	contribution
1	1/6
2	7/6
3	5/6, 3/2
4	7/3
6	1/2, 5/6
12	5/3, 2

Table 7.8: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 6)$ and $(\lambda_1, \lambda_2) = (1, \zeta_6^5)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	5/12
2	17/12
3	13/12, 3/4
4	11/6
6	3/4, 13/12
12	7/6, 5/2

Table 7.9: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 12)$ and $(\lambda_1, \lambda_2) = (1, \zeta_{12}^7)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 7/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 5/12, every singularity appearing is canonical.

d	contribution
1	$1/12$
2	$13/12$
3	$3/4,\ \ 17/12$
4	$13/6$
6	$5/12,\ \ 7/4$
12	$3/2,\ \ 11/6$

Table 7.10: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (1, 12)$ and $(\lambda_1, \lambda_2) = (1, \zeta_{12}^{11})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $11/12$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $1/12$, every singularity appearing is canonical.

d	contribution
1	1
2	0
3	$5/3,\ \ 1/3$
4	2
6	$4/3,\ \ 2/3$
12	$4/3,\ \ 8/3$

Table 7.11: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 2)$ and $(\lambda_1, \lambda_2) = (-1, -1)$

This case is similar to that of Table 7.2 and yields the same list of singularities.

d	contribution
1	7/6
2	1/6
3	5/6, 1/2
4	7/3
6	3/2, 5/6
12	5/3, 2

Table 7.12: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 3)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_3)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	5/6
2	5/6
3	3/2, 1/6
4	5/3
6	7/6, 1/2
12	2, 7/3

Table 7.13: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 3)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_3^2)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	$5/4$
2	$1/4$
3	$11/12,\ 7/12$
4	$3/2$
6	$19/12,\ 11/12$
12	$11/6,\ 13/6$

Table 7.14: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 4)$ and $(\lambda_1, \lambda_2) = (-1, i)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $3/4$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $1/4$, every singularity appearing is canonical.

d	contribution
1	$4/3$
2	$1/3$
3	$1,\ 2/3$
4	$5/3$
6	$2/3,\ 1$
12	$2,\ 7/3$

Table 7.15: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 6)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_6)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_2^{(0)} \oplus V_6^{(1)} \oplus V_2^{(0)} \oplus V_2^{(0)}$. Note that this contributes a singularity of type $\frac{1}{3}(1, 1)$ and can only occur when $n = 2$.

d	contribution
1	2/3
2	2/3
3	4/3, 1
4	7/3
6	1, 1/3
12	5/3, 2

Table 7.16: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 6)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_6^5)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_2^{(0)} \oplus V_6^{(2)} \oplus V_6^{(2)} \oplus V_6^{(2)}$. Note that this contributes a singularity of type $\frac{1}{3}(1,1)$ and can only occur when $n = 2$.

d	contribution
1	17/12
2	5/12
3	13/12, 3/4
4	11/6
6	3/4, 13/12
12	7/6, 5/2

Table 7.17: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 12)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_{12})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $7/12$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $5/12$, every singularity appearing is canonical.

d	contribution
1	13/12
2	1/12
3	4/3, 5/12
4	13/6
6	17/12, 3/4
12	3/2, 11/6

Table 7.18: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (2, 12)$ and $(\lambda_1, \lambda_2) = (-1, \zeta_{12}^5)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 11/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/12, every singularity appearing is canonical.

d	contribution
1	4/3
2	1/3
3	0, 2/3
4	8/3
6	5/3, 1
12	2, 4/3

Table 7.19: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 3)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_3)$

This case is similar to that of Table 7.2 and yields the same list of singularities.

d	contribution
1	1
2	1
3	$2/3, \;\; 1/3$
4	2
6	$4/3, \;\; 2/3$
12	$7/3, \;\; 5/3$

Table 7.20: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 3)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_3^2)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_3^{(1)} \oplus V_3^{(2)} \oplus V_3^{(2)} \oplus V_3^{(2)}$. Note that this contributes a singularity of type $\frac{1}{3}(1, 1)$ and can only occur when $n = 2$.

d	contribution
1	$2/3$
2	$5/3$
3	$4/3, \;\; 0$
4	$4/3$
6	$1, \;\; 1/3$
12	$7/3, \;\; 2$

Table 7.21: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 3)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, \zeta_3^2)$

This case is similar to that of Table 7.2 and yields the same list of singularities.

d	contribution
1	17/12
2	5/12
3	1/12, 3/4
4	11/6
6	7/4, 13/12
12	13/6, 3/2

Table 7.22: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 4)$ and $(\lambda_1, \lambda_2) = (\zeta_3, i)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 11/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/12, every singularity appearing is canonical.

d	contribution
1	13/12
2	13/12
3	3/4, 5/12
4	7/6
6	17/12, 3/4
12	5/2, 11/6

Table 7.23: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 4)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, i)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 7/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 5/12, every singularity appearing is canonical.

d	contribution
1	$3/2$
2	$1/2$
3	$1/6,\ \ 5/6$
4	2
6	$5/6,\ \ 7/6$
12	$7/3,\ \ 5/3$

Table 7.24: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_6)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	$5/6$
2	$5/6$
3	$1/2,\ \ 7/6$
4	$8/3$
6	$7/6,\ \ 1/2$
12	$2,\ \ 4/3$

Table 7.25: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_6^5)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	$7/6$
2	$7/6$
3	$5/6,\ \ 1/2$
4	$4/3$
6	$1/2,\ \ 5/6$
12	$8/3,\ \ 2$

Table 7.26: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, \zeta_6)$

This case is similar to that of Table 7.3 as all contributions are $\geq 1/2$, therefore all singularities are canonical.

d	contribution
1	$1/2$
2	$3/2$
3	$7/6,\ \ 5/6$
4	2
6	$5/6,\ \ 1/6$
12	$7/3,\ \ 5/3$

Table 7.27: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, \zeta_6^5)$

This case is similar to that of Table 7.7 and yields the same list of singularities.

d	contribution
1	19/12
2	7/12
3	1/4, 11/12
4	13/6
6	11/12, 5/4
12	3/2, 11/6

Table 7.28: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_{12})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 3/4 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/4, every singularity appearing is canonical.

d	contribution
1	3/4
2	3/4
3	5/12, 13/12
4	5/2
6	13/12, 17/12
12	11/6, 7/6

Table 7.29: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_3, \zeta_{12}^{11})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 7/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 5/12, every singularity appearing is canonical.

d	contribution
1	11/12
2	11/12
3	19/12, 1/4
4	11/6
6	5/4, 7/12
12	13/6, 3/2

Table 7.30: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, \zeta_{12}^5)$

d	contribution
1	3/4
2	7/4
3	17/12, 1/12
4	3/2
6	13/12, 5/12
12	11/6, 13/6

Table 7.31: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (3, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_3^2, \zeta_{12}^7)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 11/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/12, every singularity appearing is canonical.

d	contribution
1	$13/12$
2	$13/12$
3	$3/4,\ 17/12$
4	$7/6$
6	$5/12,\ 3/4$
12	$5/2,\ 11/6$

Table 7.32: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (4, 6)$ and $(\lambda_1, \lambda_2) = (-i, \zeta_6)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $7/12$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $5/12$, every singularity appearing is canonical.

d	contribution
1	$5/12$
2	$17/12$
3	$13/12,\ 7/4$
4	$11/6$
6	$3/4,\ 1/12$
12	$13/6,\ 3/2$

Table 7.33: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (4, 6)$ and $(\lambda_1, \lambda_2) = (-i, \zeta_6^5)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $11/12$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $1/12$, every singularity appearing is canonical.

d	contribution
1	$5/3$
2	$2/3$
3	$1/3$, 1
4	$4/3$
6	0, $4/3$
12	$8/3$, 2

Table 7.34: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_6, \zeta_6)$

This case is similar to that of Table 7.2 and yields the same list of singularities.

d	contribution
1	1
2	1
3	$2/3$, $4/3$
4	2
6	$1/3$, $2/3$
12	$7/3$, $5/3$

Table 7.35: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_6, \zeta_6^5)$

In this case, there does not exist a $V_d^{(k)}$ with trivial contribution to the Reid–Tai sum. The only case with $\sum(g) < 1$ is where $\Lambda_K \cong V_6^{(1)} \oplus V_6^{(2)} \oplus V_6^{(1)} \oplus V_6^{(1)}$. Note that this contributes a singularity of type $\frac{1}{3}(1, 1)$ and can only occur when $n = 2$.

d	contribution
1	1/3
2	4/3
3	1, 2/3
4	8/3
6	2/3, 0
12	2, 4/3

Table 7.36: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 6)$ and $(\lambda_1, \lambda_2) = (\zeta_6^5, \zeta_6^5)$

This case is similar to that of Table 7.2 and yields the same list of singularities.

d	contribution
1	7/4
2	3/4
3	5/12, 13/12
4	3/2
6	1/12, 17/12
12	11/6, 13/6

Table 7.37: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_6, \zeta_{12})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 11/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/12, every singularity appearing is canonical.

d	contribution
1	11/12
2	11/12
3	7/12, 5/4
4	11/6
6	1/4, 17/12
12	13/6, 3/2

Table 7.38: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_6, \zeta_{12}^{11})$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 3/4 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 1/4, every singularity appearing is canonical.

d	contribution
1	3/4
2	3/4
3	17/12, 13/12
4	5/2
6	1/12, 5/12
12	11/6, 7/6

Table 7.39: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_6^5, \zeta_{12}^5)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is 7/12 by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least 5/12, every singularity appearing is canonical.

d	contribution
1	$7/12$
2	$19/12$
3	$5/4,\ \ 11/12$
4	$13/6$
6	$11/12,\ \ 1/4$
12	$3/2,\ \ 11/6$

Table 7.40: Contributions of $V_d^{(k)}$ to $\sum(g)$ when $(r_1, r_2) = (6, 12)$ and $(\lambda_1, \lambda_2) = (\zeta_6^5, \zeta_{12}^7)$

In this case, we have that the contribution from $V_{r_1}^{(k_1)} \oplus V_{r_2}^{(k_2)}$ is $3/4$ by Lemma 7.2.3. By dimension restrictions, we must have at least one d component appearing in the decomposition. Since every contribution is at least $1/4$, every singularity appearing is canonical.

The following table summarizes this analysis for all possible decompositions of W. Note that if there are no trivial representations in the given case, then the dimension of the small representation must be equal to $n+2$. So for many of these cases, taking n large enough eliminates them.

(r_1, r_2)	(λ_1, λ_2)	Trivial Reps?	Small Rep	Singularity
$(1, 1)$	$(1, 1)$	Yes	$V_3^{(1)}$	$\frac{1}{3}(1, 1)$
$(1, 1)$	$(1, 1)$	Yes	$V_6^{(1)}$	$\frac{1}{6}(1, 1)$
$(1, 1)$	$(1, 1)$	Yes	$V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1, 1, 1, 1)$
$(1, 3)$	$(1, \zeta_3)$	No	$V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{3}(1, 1)$
$(1, 3)$	$(1, \zeta_3^2)$	No	$V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{3}(1, 1)$
$(1, 6)$	$(1, \zeta_6)$	No	$V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1, 1)$
$(1, 6)$	$(1, \zeta_6)$	No	$V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1, 1, 1)$
$(1, 6)$	$(1, \zeta_6)$	No	$V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1, 1, 1, 1)$
$(1, 6)$	$(1, \zeta_6)$	No	$V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1, 1, 1, 1, 1)$

$(1,6)$	$(1,\zeta_6)$	No	$V_3^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1,1,2)$
$(1,6)$	$(1,\zeta_6)$	No	$V_3^{(1)} \oplus V_6^{(1)} \oplus V_6^{(1)}$	$\frac{1}{6}(1,1,1,2)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1,1)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1,1,1)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1,1,1,1)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_6^{(1)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1,2)$
$(1,6)$	$(1,\zeta_6^5)$	No	$V_6^{(1)} \oplus V_1^{(0)} \oplus V_1^{(0)}$	$\frac{1}{6}(1,1,1,2)$
$(2,2)$	$(-1,-1)$	Yes	$V_6^{(2)}$	$\frac{1}{3}(1,1)$
$(2,2)$	$(-1,-1)$	Yes	$V_3^{(2)}$	$\frac{1}{6}(1,1)$
$(2,2)$	$(-1,-1)$	Yes	$V_3^{(2)} \oplus V_3^{(2)}$	$\frac{1}{6}(1,1,1,1)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1,1)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1,1,1)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1,1,1,1)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_3^{(2)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1,2)$
$(2,3)$	$(-1,\zeta_3)$	No	$V_3^{(2)} \oplus V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{6}(1,1,1,2)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,1)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,1,1)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_6^{(2)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1,2)$
$(2,3)$	$(-1,\zeta_3^2)$	No	$V_6^{(2)} \oplus V_3^{(1)} \oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,2)$
$(2,6)$	$(-1,\zeta_6)$	No	$V_2^{(0)} \oplus V_2^{(0)}$	$\frac{1}{3}(1,1)$
$(2,6)$	$(-1,\zeta_6^5)$	No	$V_6^{(2)} \oplus V_6^{(2)}$	$\frac{1}{3}(1,1)$

$(3,3)$	(ζ_3,ζ_3)	Yes	$V_3^{(2)}$	$\frac{1}{3}(1,1)$
$(3,3)$	(ζ_3,ζ_3)	Yes	$V_2^{(0)}$	$\frac{1}{6}(1,1)$
$(3,3)$	(ζ_3,ζ_3)	Yes	$V_2^{(0)}\oplus V_2^{(0)}$	$\frac{1}{6}(1,1,1,1)$
$(3,3)$	(ζ_3,ζ_3^2)	No	$V_3^{(2)}\oplus V_3^{(2)}$	$\frac{1}{3}(1,1)$
$(3,3)$	(ζ_3^2,ζ_3^2)	Yes	$V_1^{(0)}$	$\frac{1}{3}(1,1)$
$(3,3)$	(ζ_3^2,ζ_3^2)	Yes	$V_6^{(2)}$	$\frac{1}{6}(1,1)$
$(3,3)$	(ζ_3^2,ζ_3^2)	Yes	$V_6^{(2)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1,1,1)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,1)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,1,1)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_2^{(0)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,2)$
$(3,6)$	(ζ_3,ζ_6)	No	$V_2^{(0)}\oplus V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,2)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_6^{(2)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_6^{(2)}\oplus V_6^{(2)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_6^{(2)}\oplus V_6^{(2)}\oplus V_6^{(2)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1,1,1)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_6^{(2)}\oplus V_6^{(2)}\oplus V_6^{(2)}\oplus V_6^{(2)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1,1,1,1)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_1^{(0)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1,2)$
$(3,6)$	(ζ_3^2,ζ_6^5)	No	$V_1^{(0)}\oplus V_6^{(2)}\oplus V_6^{(2)}$	$\frac{1}{6}(1,1,1,2)$
$(6,6)$	(ζ_6,ζ_6)	Yes	$V_2^{(0)}$	$\frac{1}{3}(1,1)$
$(6,6)$	(ζ_6,ζ_6)	Yes	$V_3^{(1)}$	$\frac{1}{6}(1,1)$
$(6,6)$	(ζ_6,ζ_6)	Yes	$V_3^{(1)}\oplus V_3^{(1)}$	$\frac{1}{6}(1,1,1,1)$
$(6,6)$	(ζ_6,ζ_6^5)	No	$V_6^{(1)}\oplus V_6^{(1)}$	$\frac{1}{3}(1,1)$
$(6,6)$	(ζ_6^5,ζ_6^5)	Yes	$V_6^{(1)}$	$\frac{1}{3}(1,1)$
$(6,6)$	(ζ_6^5,ζ_6^5)	Yes	$V_1^{(0)}$	$\frac{1}{6}(1,1)$

$(6,6)$	(ζ_6^5, ζ_6^5)	Yes	$V_1^{(0)} \oplus V_6^{(2)}$	$\frac{1}{6}(1,1,1,1)$

$\square$

We must now consider the case where $W \subseteq V_r^{(k)}$. The following proof is of a similar flavor to Lemma 4.3.1.

Lemma 7.2.5. *Suppose W is contained in the complexification of one irreducible component of the decomposition of Λ_K as a g-module. Then either $\sum(g) \geq 1$ or $W \cong V_4^{(0)}$, $W \cong V_{12}^{(1)}$, or $W \cong V_{12}^{(2)}$.*

Proof. By computer calculation, we can verify that for all $r \leq 120$ the minimal contribution of $V_r^{(k)}$ to $\sum(g)$ is only < 1 if $r = 4$ or $r = 12$. So if we can show that $\sum(g) \geq 1$ for $r > 120$, then we are done.

As in the proof of Lemma 4.3.1, we proceed in two cases, either $3 \mid r$ or $3 \nmid r$. Suppose first that $3 \mid r$, then

$$\sum(g) \geq 2 \sum_{k=1}^{\frac{\varphi(r)}{2}-2} \frac{k}{r}.$$

Note that $\varphi(r) \neq 2$ since then $\dim V_r^{(k)} = 1$. If $\varphi(r) \geq 8$, then

$$\begin{aligned}
2 \sum_{k=1}^{\frac{\varphi(r)}{2}-2} \frac{k}{r} &= \sum_{k=1}^{\frac{\varphi(r)}{2}-2} \frac{k}{r} + \frac{\left(\frac{\varphi(r)}{2} - 2\right)\left(\frac{\varphi(r)}{2} - 1\right)}{2} \\
&\geq \sum_{k=1}^{\frac{\varphi(r)}{2}-2} \frac{k}{r} + \left(\frac{\varphi(r)}{2} - 1\right) 2 \\
&= \sum_{k=1}^{\frac{\varphi(r)}{2}-1} \frac{k}{r}.
\end{aligned}$$

Therefore by the proof of Lemma 4.3.1, we have that if $3 \nmid r$ and $r \neq 3, 6, 9, 12, 15, 18, 21,$ $24, 30,$ $36, 42, 48,$ $54, 60, 66,$ $72, 78, 84, 90, 120$, then $\sum(g) \geq 1$. Note that this also includes all cases where $3 \mid r$ and $2 < \varphi(r) \leq 6$. Since $r \leq 120$ in all cases, we are done by our computer calculation.

Now we assume $3 \nmid r$. So,

$$\sum(g) \geq 2 \sum_{k=1}^{\varphi(r)-2} \frac{k}{r}.$$

By the same logic as the previous case, if $\varphi(r) \geq 4$, then

$$2 \sum_{k=1}^{\varphi(r)-2} \frac{k}{r} \geq \sum_{k=1}^{\varphi(r)-1} \frac{k}{r}.$$

By the proof of Lemma 4.3.1, we have that in this case if $r \neq 1, 2, 4, 8, 10$, then $\sum(g) \geq 1$. Since all of these cases have $r \leq 120$, we are done by our computer calculation. $\square$

Theorem 7.2.6. *If $n \geq 6$, then the singularities of $\mathbb{B}_\Lambda(2, n)/\Gamma$ are either canonical, or, each non-canonical singularity is locally given as a quotient $\mathbb{C}^{2n}/G$, where the finite group G admits a cyclic subgroup $\langle g \rangle \subseteq G$ so that $\mathbb{C}^{2n}/\langle g \rangle$ is of type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, or $\frac{1}{6}(1,1,1,1)$. When $n = 2$, there may also be singularities of type $\frac{1}{6}(1,1,2)$. When $n = 3$, there may also be singularities of type $\frac{1}{6}(1,1,1)$ or $\frac{1}{6}(1,1,1,2)$. When $n = 5$, there may also be singularities of type $\frac{1}{6}(1,1,1,1,1)$.*

Proof. By Lemma 7.2.4 and Lemma 7.2.5, it suffices to show that if $W \cong V_4^{(0)}$, $W \cong V_{12}^{(1)}$, or $W \cong V_{12}^{(2)}$, then $\sum(g) \geq 1$.

We begin by considering $W \cong V_4^{(0)}$. In this case the eigenvalues of W are i and $-i$. If some V_1 appears in the decomposition of $\Lambda_\mathbb{C}$, then on the tangent space we get two eigenvalues by Lemma 10.1, $i^{-1} \cdot 1 = i$ and $(-i)^{-1} \cdot 1 = -i$. But these eigenvalues are complex conjugate, so we get a contribution of 1 to the Reid–Tai sum. The case of V_2 is essentially identical. If some $V_3^{(1)}$ appears, we get eigenvalues $i \cdot \zeta_3 = \zeta_{12}^7$ and $-i \cdot \zeta_3 = \zeta_{12}$, so we get a contribution of $2/3$. If $V_3^{(2)}$ appears, we get eigenvalues $i \cdot \zeta_3^2 = \zeta_{12}^{11}$ and $-i \cdot \zeta_3^2 = \zeta_{12}^5$, so we get a contribution of $16/12$. If $V_4^{(0)}$ appears, then we get 4 known eigenvalues on the tangent space: $1, 1, -1, -1$, so we get a contribution of 1. If $V_6^{(1)}$ appears, then we have eigenvalues $i \cdot \zeta_6 = \zeta_{12}^5$ and $-i \cdot \zeta_6 = \zeta_{12}^{11}$, so we get a contribution of $16/12$. If $V_6^{(2)}$ appears, we have eigenvalues $i \cdot \zeta_6^5 = \zeta_{12}$ and $-i \cdot \zeta_6^5 = \zeta_{12}^7$ so we get a contribution of $2/3$. For $V_{12}^{(1)}$, we have 4 known eigenvalues: $i\zeta_{12}, -i\zeta_{12}, i\zeta_{12}^7, -i\zeta_{12}^7$, this gives a Reid–Tai contribution of $28/12$. For $V_{12}^{(2)}$, we have 4 known eigenvalues: $i\zeta_{12}^5, -i\zeta_{12}^5, i\zeta_{12}^{11}, -i\zeta_{12}^{11}$ with a total Reid–Tai contribution of $20/12$.

We summarize the above computation in Table 7.42.

d	Contribution
1	1
2	1
3	2/3, 4/3
4	1
6	4/3, 2/3
12	7/3, 5/3

Table 7.42: Contributions of $V_d^{(k)}$ for $W \cong V_4^{(0)}$.

So the only way we get $\sum(g) \leq 1$ is if we only have one $V_3^{(1)}$ or one $V_6^{(2)}$ appearing in the decomposition of Λ_K. But since we are in the case of the generalized ball, we have that Λ_K has dimension at least 4 and $\dim V_4^{(0)} = 2$, $\dim V_3^{(1)} = 1$, $\dim V_6^{(2)} = 1$. Therefore, if $W \cong V_4^{(0)}$, we have that $\sum(g) \geq 1$.

Carrying out the same computation for $W \cong V_{12}^{(1)}$, we obtain the following table:

d	Contribution
1	4/3
2	4/3
3	1, 2/3
4	5/3
6	2/3, 1
12	1, 7/3

Table 7.43: Contributions of $V_d^{(k)}$ for $W \cong V_{12}^{(1)}$.

Carrying out the same computation for $W \cong V_{12}^{(2)}$, we obtain the following table:

d	Contribution
1	1
2	1
3	2/3, 4/3
4	1
6	4/3, 2/3
12	7/3, 5/3

Table 7.44: Contributions of $V_d^{(k)}$ for $W \cong V_{12}^{(2)}$.

□

Corollary 7.2.7. *The generalized ball quotient $\mathbb{B}_\Lambda(2,2)/\Gamma$ defined by $\mathbb{Q}(\zeta_3)$ has noncanonical singularities if and only if there exists $g \in \Gamma$ such that $\mathbb{B}_\Lambda(2,2)/\langle g \rangle$ has a singularity of either type $\frac{1}{3}(1,1)$, $\frac{1}{6}(1,1)$, or $\frac{1}{6}(1,1,2)$.*

Proof. There are no quasi-reflections in Γ and these are the only possible noncanonical cyclic quotients in the $n = 2$ case. □

Remark 7.2.8. These methods generalize to generalized ball quotients defined by other imaginary quadratic number fields. The representation theory changes slightly, but most of the necessary details can be found in [Beh11]. We predict that a similar result to [Beh11, Corollary 2] can be proven, i.e., the singularities of $\mathbb{B}_\Lambda(2,n)/\Gamma$ are canonical for n large with Λ defined by $\mathbb{Q}(\sqrt{-D})$, $D > 3$.

Remark 7.2.9. In another direction, if we consider generalized ball quotients $\mathbb{B}_\Lambda(m,n)/\Gamma$, we predict that for m large enough the singularities of $\mathbb{B}_\Lambda(m,n)/\Gamma$ will be canonical for all $n \geq m$, where Λ is defined by any imaginary quadratic number field.

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Appendix A

The Reid–Tai Criterion

In this appendix we explore results related to the Reid–Tai criterion. In Behrens' paper, the Reid–Tai criterion is stated as follows:

Theorem A.0.1 (Reid–Tai criterion, [Beh11])**.** *Let H be a finite subgroup of* $\mathrm{GL}(\mathbb{C}^k)$ *without quasi-reflections. Then $\mathbb{C}^k/H$ has canonical singularities if and only if*

$$\sum(A) \geq 1$$

for every $A \in H$, $A \neq I$.

We are interested in understanding what can be said in the case where H acts with quasi-reflections and what breaks down in the argument. We start with a smooth variety X and a finite subgroup $G \subseteq \mathrm{Aut}(X)$ acting on X. We want a criterion for when the quotient X/G has canonical singularities. The approach to developing this criteria can be made in two steps:

(1) Reduce to the case of cyclic subgroups of G.

(2) Find a condition for when $X/\langle g\rangle$ is canonical.

We begin addressing (1) with the following theorem.

Theorem 1A. *Suppose X is smooth. If X/G has non-canonical singularities, then there exists some $g \in G$ such that $X/\langle g\rangle$ is non-canonical.*

If G acts without quasi-reflections, then we have the stronger theorem:

Theorem 1B. *If G acts without quasi-reflections and X is smooth, then X/G is canonical if and only if $X/\langle g\rangle$ is canonical for all $g \in G$.*

Theorem 1A will follow as a consequence of Theorem 2A. We will prove Theorem 1B now by applying [Tai82, Proposition 3.1].

Let Γ be a finite group acting on $\mathbb{C}^N$, $X = \mathbb{C}^N/\Gamma$, and $X_g = \mathbb{C}^N/\langle g\rangle$ for any $g \in \Gamma$. Let $\widetilde{X}, \widetilde{X_g}$ be smooth models for X and X_g respectively.

Proposition A.0.2. *If η is a pluri-canonical form on $\mathbb{C}^N$ invariant under Γ, then η extends to $\widetilde{X}$ if and only if η extends to $\widetilde{X_g}$ for every $g \in \Gamma$.*

Proof. [Tai82, Proposition 3.1] □

Every pluri-canonical form on X extends to a pluri-canonical form on the resolution $\widetilde{X}$ if and only if X has canonical singularities. Indeed this implies directly that

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum a_i E_i$$

with $a_i \geq 0$.

Since $\sum a_i E_i$ is effective, we have a short exact sequence

$$0 \to f^*rK_X \to f^*rK_X + \sum a_i E_i \to \mathcal{Q} \to 0.$$

So X having canonical singularities implies that every pluricanonical form on X extends to $\widetilde{X}$. Conversely, the existence of an extension for every form implies that $\sum a_i E_i$ is effective.

Lemma A.0.3. *Suppose X is smooth and G is a finite group acting without quasi-reflections. Then there exists $r > 0$ such that every form in $rK_{X/G}$ is induced by a G-invariant form in rK_X.*

Proof. Note that X/G has finite quotient singularities, so X/G is $\mathbb{Q}$-Gorenstein (quotient implies log canonical implies $\mathbb{Q}$-Gorenstein). Therefore $rK_{X/G}$ is Cartier for some $r > 0$. The quotient map $\pi : X \to X/G$ is finite with $R_\pi = 0$ since the ramification divisor of a finite quotient map is given by the fixed locus of the quasi-reflections. In general we have that $rK_X = \pi^* rK_{X/G} + rR_\pi$, so in particular we have that $rK_X = \pi^* rK_{X/G}$. So for any form $s \in rK_{X/G}$, we have that $\pi^* s \in rK_X$ is a G-invariant form which induces s. $\square$

We now prove Theorem 1B.

Proof of 1B. Since X is smooth we can immediately reduce to case where $X = \mathbb{C}^N$ by studying the action on the tangent space of X at a fixed point under Γ. Let $Y_g = \mathbb{C}^N/\langle g \rangle$ and $Y = \mathbb{C}^N/G$. Let $\widetilde{Y_g}$ and $\widetilde{Y}$ be smooth models for Y_g and Y respectively.

Suppose for each $g \in G$, $Y_g = \mathbb{C}^N/\langle g \rangle$ is canonical.

Let η be a pluri-canonical form on $\mathbb{C}^N$ invariant under G. Since Y_g has canonical singularities, η extends to $\widetilde{Y_g}$ for all $g \in G$. By [Tai82, Proposition 3.1], η extends to $\widetilde{Y}$. By the previous lemma, every pluri-canonical form on Y is induced by some pluri-canonical η on $\mathbb{C}^N$ invariant under G. So every pluri-canonical form on Y extends to $\widetilde{Y}$. Thus Y has canonical singularities.

Conversely, if Y has canonical singularities then every η extends to $\widetilde{Y}$. By [Tai82, Proposition 3.1], every η extends to $\widetilde{Y_g}$ for every $g \in G$. By the previous lemma, every pluri-canonical form on Y_g is induced by some η. Therefore every pluri-canonical form on Y_g extends to $\widetilde{Y_g}$ so each Y_g has canonical singularities. $\square$

If G acts with quasi-reflections, then we cannot apply this argument with Tai's proposition. In general we have that $K_{\mathbb{C}^N} = \pi^* K_{\mathbb{C}^N/G} + R$ where R is the ramification divisor of the quotient map $\pi : \mathbb{C}^N \to \mathbb{C}^N/G$. If G acts without quasi-reflections, then $R = 0$ so there is a one to one correspondence between invariant forms on $\mathbb{C}^N$ and forms on $\mathbb{C}^N/G$. If G acts with quasi-reflections, then $R > 0$.

The converse to Theorem 1A is not true in general. Consider $X = \mathbb{C}^2$ and $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with G acting on X via $(1,0) \cdot (x,y) = (\zeta_3 x, y)$ and $(0,1) \cdot (x,y) = (x, \zeta_3 y)$. Let $D = \langle (1,1) \rangle \cong \mathbb{Z}_3$ be the cyclic subgroup of G acting on $\mathbb{C}^2$ via $(1,1) \cdot (x,y) = (\zeta_3 x, \zeta_3 y)$. Then X/D has ring of functions $\mathbb{C}[x^3, x^2y, xy^2, y^3]$ and is the cone over the twisted cubic. So X/D has a non-canonical singularity. But X/G is isomorphic to $\mathbb{C}^2$ which is smooth and therefore canonical. So in general if some cyclic subgroup gives non-canonical singularities, we can conclude nothing about the full quotient X/G.

A.1 A Statement for Normal Varieties

For the rest of this appendix, let k be an algebraically closed field of characteristic 0. Typically the Reid-Tai Criterion is applied to finite quotients of smooth varieties. We will consider finite groups acting on normal schemes of finite type over k.

Theorem 2A. *Let G be a finite group acting on a normal scheme V of finite type over k. If V/G has non-canonical singularities, then there exists some $g \in G$ such that $V/\langle g\rangle$ is non-canonical.*

In the case where V was nonsingular, we used local coordinates to define when g was a quasi-reflection. Now we use the idea that a quasi-reflection should have a fixed locus of codimension 1.

Definition A.1.1. Let G be a finite group acting on a normal scheme V. We say that $g \in G$ is a quasi-reflection if the quotient morphism $V \to V/\langle g\rangle$ is ramified in codimension 1.

Therefore if G acts without quasi-reflections, $V \to V/G$ is unramified in codimension 1. We now state a generalization of Theorem 1B.

Theorem 2B. *Let G be a finite group acting without quasi-reflections on a normal scheme V of finite type over k. If $V/\langle g\rangle$ has canonical singularities for all $g \in G$, then V/G has canonical singularities. If V has at worst log canonical singularities and V/G has canonical singularities, then $V/\langle g\rangle$ has canonical singularities for all $g \in G$.*

A.2 Diagrams

We construct two diagrams that we use to prove Theorems 2A and 2B.

Suppose G is a finite group acting linearly on V, a normal scheme of finite type over k. Let $X = V/G$ and $\widetilde{X} \to X$ be a resolution of singularities and $E \subseteq \widetilde{X}$ be a prime divisor. We let $p : \widetilde{V} \to \widetilde{X}$ be the normalization of $\widetilde{X}$ in the field of fractions of V. Recall that p is a finite morphism [Har77, Exercise II.3.8]. Note that $\widetilde{V}$ is birational to V. The group G acts on $\widetilde{V}$ and $\widetilde{X}$. In fact, $\widetilde{X} = \widetilde{V}/G$.

Let $F \subseteq \widetilde{V}$ be a prime divisor dominating E. Let C_F be the subgroup of G acting trivially on F. The subgroup C_F is necessarily cyclic since $\widetilde{V}$ is generically smooth along F. Note that $\widetilde{V}/C_F$ is a resolution of V/C_F. Moreover, the quotient map $q : \widetilde{V} \to \widetilde{V}/C_F$ is a finite morphism. Define $E' = q(F)$. We have that E' is a prime exceptional divisor of $\widetilde{V}/C_F \to V/C_F$.

This all can be summarized in the following lemma.

Lemma A.2.1 (Lifting to a cyclic quotient)**.** *Let G be a finite group acting on a normal scheme V of finite type over k. Let $X = V/G$ and $\widetilde{X} \to X$ be a resolution of singularities of X with $E \subseteq \widetilde{X}$ a prime exceptional divisor. Let $\widetilde{V}$ be the normalization of $\widetilde{X}$ in the field of fractions of V and $F \subseteq \widetilde{V}$ be a prime divisor dominating E. Then there exists a cyclic subgroup C_F of G such that F dominates $E' = q(F) \subseteq \widetilde{V}/C_F$ and E' dominates E. Moreover we get a commutative diagram*

$$\begin{tikzcd}
F \arrow[r, hook] & \widetilde{V} \arrow[r] \arrow[d, "q"'] & V \arrow[d] \\
E' \arrow[r, hook] & \widetilde{V}/C_F \arrow[r] \arrow[d] & V/C_F \arrow[d] \\
E \arrow[r, hook] & \widetilde{X} \arrow[r] & X
\end{tikzcd}$$

where the vertical morphisms are finite and the horizontal morphisms are birational.

Note that due to the discussion after Definition 2.1 in [Kol13], considering the discrepancy on the normalization is the same as considering the discrepancy on some resolution of V. We can think of this lemma as saying that exceptional divisors on the full quotient can be lifted to related exceptional divisors on some partial quotient by a cyclic subgroup.

The goal of our next construction is to start with an exceptional divisor of some cyclic quotient and construct a related exceptional divisor on the full quotient. We will use the following technical tool in our construction:

Lemma A.2.2 (Zariski-Abhyankar)**.** *Let $f : Y \to X$ be a dominant morphism of finite type of schemes. Let $y \in Y$ be a codimension 1 regular point that does not dominate any of the irreducible components of X.*

By blowing up finitely many times, we construct a rational map $f_n : Y \dashrightarrow X_n$ such that $x_n := f_n(y)$ is a codimension 1 regular point of X_n.

Proof. [Kol13, Lemma 2.22] $\square$

We always have

$$\begin{tikzcd} V \arrow[dd, "\pi_G"'] \arrow[dr, "\pi_C"] & \\ & V/C \arrow[dl] \\ X & \end{tikzcd}$$

where the map $V/C \to X$ is induced by the inclusion of $K(V)^G \to K(V)^C$. Let $\widetilde{V/C}$ be a resolution of V/C with exceptional divisor E'. By the result of Zariski and Abhyankar, we can construct a partial resolution $\widetilde{X} \to X$ such that the exceptional divisor E is dominated by E'. So we have constructed the following diagram:

$$\begin{tikzcd} & & V \arrow[d] \\ E' \arrow[r, hook] \arrow[d, dashed] & \widetilde{V/C} \arrow[r] \arrow[d, dashed] & V/C \arrow[d] \\ E \arrow[r, hook] & \widetilde{X} \arrow[r] & X \end{tikzcd}$$

Now we define $\widetilde{\widetilde{V/C}}$ to be the normalization of $\widetilde{X}$ in the field of fractions of V/C. This gives a new divisor E'' in $\widetilde{\widetilde{V/C}}$. We have the following properties:

(1) $\widetilde{\widetilde{V/C}} \to \widetilde{X}$ is a finite morphism.

(2) $E'' \to E$ is a morphism.

(3) $(\widetilde{\widetilde{V/C}}, E'') \dashrightarrow (\widetilde{V/C}, E')$ is an isomorphism at the generic points of E'' and E'.

Property (3) implies $a(E'', V/C) = a(E', V/C)$, so this construction still allows us to compare discrepancies. The point of taking the normalization is that we have now upgraded our previous diagram so that the vertical maps are finite morphisms. So now we have:

$$\begin{array}{ccccc} & & & & V \\ & & & & \downarrow \\ E'' & \hookrightarrow & \widetilde{\widetilde{V/C}} & \longrightarrow & V/C \\ \downarrow & & \downarrow{\scriptstyle \text{finite}} & & \downarrow{\scriptstyle \text{finite}} \\ E & \hookrightarrow & \widetilde{X} & \longrightarrow & X \end{array}$$

Now we define $\widetilde{V}$ to be the integral closure of $\widetilde{\widetilde{V/C}}$ in $K(V)$. We get an exceptional divisor F of $\widetilde{V} \to V$. By the same argument as before, $\widetilde{V} \to \widetilde{\widetilde{V/C}}$ is a finite morphism and $F \to E''$ is a dominant morphism.

This construction is summarized in the following lemma.

Lemma A.2.3 (Descending from a cyclic quotient)**.** *Let G be a finite group acting on a normal scheme V of finite type over k. Let $X = V/G$, $C \leq G$ a cyclic subgroup, and $\widetilde{V/C} \to V/C$ a resolution with exceptional divisor E'. Then there exists birational morphisms $\widetilde{X} \to X$, $\widetilde{\widetilde{V/C}} \to V/C$, and $\widetilde{V} \to V$ with respective exceptional divisors E, E'', and F such that we have the following commutative diagram*

$$\begin{array}{ccccc} F & \hookrightarrow & \widetilde{V} & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle \text{finite}} & & \downarrow{\scriptstyle \text{finite}} \\ E'' & \hookrightarrow & \widetilde{\widetilde{V/C}} & \longrightarrow & V/C \\ \downarrow & & \downarrow{\scriptstyle \text{finite}} & & \downarrow{\scriptstyle \text{finite}} \\ E & \hookrightarrow & \widetilde{X} & \longrightarrow & X \end{array}$$

where F dominates E'', E'' dominates E, $a(E'', V/C) = a(E', V/C)$, and the vertical maps are finite morphisms.

We can think of this lemma as saying that exceptional divisors on a cyclic subquotient have some associated exceptional divisor on the full quotient related via finite morphisms.

A.3 Proof of Theorem 2B

We compare discrepancies using both of the previous constructions and the following lemma

Lemma A.3.1. *Let $g : X' \to X$ be a ramified cover of degree m of normal schemes of finite type over k such that $K_{X'} = g^*K_X$. Let $f : Y \to X$ be a birational morphism with exceptional divisor E*

such that Y is normal at the generic point of Y. Let Y' be the normalization of Y in the function field of X' and $F \subset Y$ a divisor dominating E. Then,

$$a(E, X) + 1 = \frac{a(F, X') + 1}{r(F)},$$

where $r(F)$ is the ramification index of g along F.

Proof. [Kol13, Equation 2.42.4], noting that $\mathrm{char}(k) = 0$ implies $R(F)+1 = r(F)$ and we are taking $\Delta = \Delta' = 0$. □

Note that both of our previous constructions are setup so we can apply this lemma. The only potential problem is satisfying the assumption that $K_{X'} = g^*K_X$. If G acts without quasi-reflections, then this is the case since $\pi : V \to X$ has no ramification in codimension 1, so $K_V = \pi^*K_X$.

We are now ready to prove Theorem 2B.

Proof of Theorem 2B. Let $X = V/G$. Suppose that X does not have canonical singularities. So we let $\widetilde{X} \to X$ be a resolution of singularities with exceptional divisor E such that $a(E, X) < 0$. We then apply Lemma 1.9 to obtain

$$\begin{tikzcd} F \arrow[r, hook] & \widetilde{V} \arrow[r] \arrow[d, "q"'] & V \arrow[d] \\ E' \arrow[r, hook] & \widetilde{V}/C_F \arrow[r] \arrow[d] & V/C_F \arrow[d] \\ E \arrow[r, hook] & \widetilde{X} \arrow[r] & X \end{tikzcd}$$

By applying Lemma 1.12 to the full quotient

$$\begin{tikzcd} F \arrow[r, hook] & \widetilde{V} \arrow[r] \arrow[d] & V \arrow[d] \\ E \arrow[r, hook] & \widetilde{X} \arrow[r] & X \end{tikzcd}$$

we obtain

$$a(E, X) + 1 = \frac{a(F, V) + 1}{|C_F|}.$$

Note that $|C_F| = r(F)$ since C_F by definition is the subgroup of G acting trivially on F.

By applying Lemma 1.12 to the cyclic quotient

$$\begin{tikzcd}
F \arrow[r, hook] & \widetilde{V} \arrow[r] \arrow[d] & V \arrow[d] \\
E' \arrow[r, hook] & \widetilde{V}/C_F \arrow[r] & V/C_F
\end{tikzcd}$$

we obtain

$$a(E', V/C_F) + 1 = \frac{a(F,V)+1}{|C_F|}.$$

Therefore $a(E', V/C_F) = a(E,X) < 0$, so V/C_F does not have canonical singularities.

Now we suppose that V/C does not have canonical singularities for some cyclic subgroup $C \leq G$. Let $\widetilde{V/C} \to V/C$ be a resolution with exceptional divisor E' such that $a(E', V/C) < 0$. By Lemma 1.11 we construct the following diagram

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[d] & \widetilde{V} \arrow[r] \arrow[d, "\text{finite}"] & V \arrow[d, "\text{finite}"] \\
E'' \arrow[r, hook] \arrow[d] & \widetilde{\widetilde{V/C}} \arrow[r] \arrow[d, "\text{finite}"] & V/C \arrow[d, "\text{finite}"] \\
E \arrow[r, hook] & \widetilde{X} \arrow[r] & X
\end{tikzcd}$$

where $a(E'', V/C) < 0$.

As before, we apply Lemma 1.12 to the full quotient to obtain

$$a(E,X) + 1 = \frac{a(F,V)+1}{|G_F|},$$

where G_F is the subgroup of G acting trivially on F.

By applying Lemma 1.12 to the cyclic quotient we obtain

$$a(E'', V/C) + 1 = \frac{a(F,V)+1}{|C_F|},$$

where C_F is the subgroup of C acting trivially on F.

Note that $|C_F| \leq |G_F|$, therefore

$$\frac{a(F,V)+1}{|G_F|} \leq \frac{a(F,V)+1}{|C_F|} \Leftrightarrow a(F,V) \geq -1.$$

By assumption, V has at worst log canonical singularities, so $a(F,V) \geq -1$. It follows that $a(E,X) \leq a(E'', V/C) < 0$, so X does not have canonical singularities. $\square$

A.4 Proof of Theorem 2A

Now we consider the case where we allow G to act with quasi-reflections. So we can no longer apply Kollar's formula 2.42.4 since $K_V = \pi^* K_X + R_\pi$. The strategy is to use Lemma 1.9 and a more general comparison of the discrepancies involving the ramification divisors.

This boils down to the discrepancy computation of a general quotient. First we give some setup. Let X be a normal scheme and $Y = X/G$. Then we have the quotient map $q : X \to Y$. Let R_i be the ramification divisor of the map q. Let $\widetilde{Y} = \widetilde{X}/G$ be a resolution of Y and E be an exceptional divisor of $\widetilde{Y} \to Y$. Then define $\widetilde{X}$ to be the normalization of $\widetilde{Y}$ in the function field of X. Let F be a prime exceptional divisor of $\widetilde{X} \to X$ dominating E and let $\overline{R_i}$ be the strict transform of R_i in $\widetilde{X}$. This setup can be summarized in the following diagram:

$$\begin{tikzcd}
 & \widetilde{R_i} \arrow[d, hook] & R_i \arrow[d, hook] \\
F \arrow[r, hook] \arrow[d] & \widetilde{X} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{q}"] & X \arrow[d, "q"] \\
E \arrow[r, hook] & \widetilde{Y} \arrow[r, "\pi"] & Y
\end{tikzcd}$$

By [CMGHL23, (6.1)], we have

$$K_{\widetilde{X}} = \widetilde{q}^* K_{\widetilde{Y}} + \sum \left(|G_{R_i}| - 1\right) \widetilde{R_i} + (|G_F| - 1)F + \cdots ,$$

where $|G_{R_i}| - 1$ is the ramification index in terms of the stabilizer G_{R_i} of the ramification divisor and G_F is the stabilizer of F in G. We are avoiding writing down the contributions of the other components of the exceptional divisor as these will carry no weight in comparing the discrepancy of E to the discrepancy of F.

By the definition of discrepancy, we also have that

$$K_{\widetilde{X}} = \widetilde{\pi}^* K_X + a(F, X)F + \cdots .$$

In the first equation, we use the fact that $K_{\widetilde{Y}} = \pi^* K_Y + a(E, Y)E + \cdots$. Therefore,

$$K_{\widetilde{X}} = \widetilde{q}^* (\pi^* K_Y + a(E, Y)E + \cdots) + \sum \left(|G_{R_i}| - 1\right) \widetilde{R_i} + (|G_F| - 1)F + \cdots$$

$$
\begin{aligned}
&= \widetilde{q}^*\pi^*K_Y + a(E,Y)|G_F|F + \sum \left(|G_{R_i}|-1\right)\widetilde{R_i} + (|G_F|-1)F + \cdots \\
&= \widetilde{q}^*\pi^*K_Y + \sum \left(|G_{R_i}|-1\right)\widetilde{R_i} + \left(\left(a(E,Y)+1\right)|G_F|-1\right)F + \cdots
\end{aligned}
$$

In the second equation we apply the Riemann–Hurwitz formula

$$
K_X = q^*K_Y + \sum \left(|G_{R_i}|-1\right)R_i,
$$

so that

$$
\begin{aligned}
K_{\widetilde{X}} &= \widetilde{\pi}^*(q^*K_Y + \sum \left(|G_{R_i}|-1\right)R_i) + a(F,X)F + \cdots \\
&= \widetilde{\pi}^*q^*K_Y + \sum \left(|G_{R_i}-1|\right)\widetilde{R_i} + \left(\sum(|G_{R_i}|-1)\mu_i\right)F + a(F,X)F + \cdots \\
&= \widetilde{\pi}^*q^*K_Y + \sum \left(|G_{R_i}-1|\right)\widetilde{R_i} + \left(a(F,X) + \sum(|G_{R_i}|-1)\mu_i\right)F + \cdots
\end{aligned}
$$

where the μ_i are defined via $\widetilde{\pi}^*R_i = \widetilde{R_i} + \mu_i F$. In particular, each $\mu_i \geq 0$.

Comparing the coefficients of F, we have that

$$
(a(E,Y)+1)|G_F| - 1 = a(F,X) + \sum \left(|G_{R_i}|-1\right)\mu_i.
$$

So we have proven the following result:

Proposition A.4.1. *Let G be a finite group acting on X be a normal scheme of finite type over k and let $Y = X/G$. Let $\pi : \widetilde{Y} \to Y$ be a resolution and E an exceptional prime divisor. Let $\widetilde{X}$ be the normalization of $\widetilde{Y}$ in the function field of X and F an exceptional prime divisor of $\widetilde{X} \to X$ dominating E. Then*

$$
a(E,Y) = \frac{a(F,X) + 1 + \sum \left(|G_{R_i}|-1\right)\mu_i}{|G_F|} - 1, \tag{A.4.1}
$$

where G_{R_i} are the stabilizers of the ramification divisors R_i of the quotient map $q : X \to Y$, G_F is the stabilizer of F, and each $\mu_i \geq 0$ is determined by $\widetilde{\pi}^*R_i = \widetilde{R_i} + \mu_i F$.

Note that if G acts without quasi-reflections, then the quotient map is unramified in codimension 1. In other words, there are no R_i so this computation yields [Kol13, 2.42.4]. Moreover, this result is a corollary of [CMGHL23, (6.2)] with $c = a(F,X)+1$.

Equipped with this result, we are ready to prove Theorem 2A.

Proof of Theorem 2A. Suppose $X = V/G$ does not have canonical singularities. Let $\widetilde{X} \to X$ be a resolution with exceptional divisor E such that $a(E, X) < 0$. By applying Lemma 1.9, we get the following diagram

$$\begin{tikzcd} F \arrow[r, hook] & \widetilde{V} \arrow[r] \arrow[d] & V \arrow[d] \\ E' \arrow[r, hook] & \widetilde{V}/C_F \arrow[r] \arrow[d] & V/C_F \arrow[d] \\ E \arrow[r, hook] & \widetilde{X} \arrow[r] & X \end{tikzcd}$$

where C_F is the cyclic subgroup of G stabilizing F (i.e. $C_F = G_F$). Further note that this construction allows us to apply Proposition 1.13 to both the full quotient from V and the cyclic quotient from V. By applying the proposition to the full quotient, we have

$$a(E, X) = \frac{a(F, V) + 1 + \sum(|G_{R_i}| - 1)\mu_i}{|G_F|} - 1.$$

By applying the proposition to the cyclic quotient, we have

$$a(E', V/C_F) = \frac{a(F, V) + 1 + \sum\left(\left|(C_F)_{R'_j}\right| - 1\right)\mu'_j}{|C_F|} - 1.$$

Note that the ramification divisors R'_i of the cyclic quotient map $V \to V/C_F$ correspond to quasi-reflections in C_F. Any quasi-reflection in C_F is necessarily contained in G, so the divisors R'_j are a subset of the R_j. Immediately we have that $\mu_i = \mu'_i$. Indeed, the definition of μ_i only depends on how the ramification divisors pull back via $\widetilde{\pi}$. Moreover we can take our sum in the cyclic quotient to be over all R_i from the full quotient since if R_i is not a ramification divisor of $V \to V/C_F$, then $|(C_F)_{R_i}| = 1$. Therefore we have that

$$a(E', V/C_F) = \frac{a(F, V) + \sum\left(|(C_F)_{R_i}| - 1\right)\mu_i + 1}{|G_F|} - 1.$$

Since $C_F \leq G$, we have that $|(C_F)_{R_i}| \leq |G_{R_i}|$.

Therefore $a(E', V/C_F) \leq a(E, X) < 0$. So V/C_F does not have canonical singularities. $\square$

Appendix B

Code

This Appendix contains the code used for proving various results throughout the thesis. The code is written in Python and carries out various computations throughout Chapter 4 and Chapter 7.[1] We also include Macaulay2 code for computing the toric resolutions described in the proof of Lemma 6.1.4.

B.1 Python Code

```
def gen_primes():
    D = {}
    q = 2
    while True:
        if q not in D:
            yield q
            D[q * q] = [q]
        else:
            for p in D[q]:
                D.setdefault(p + q, []).append(p)
            del D[q]
        q += 1
```

[1] Warning: The code is included for the sake of completeness. This code was not written to be easy to read or parse.

```
def coprimes(d):
    k = 1
    while k <= d:
        if math.gcd(k,d) == 1:
            yield k
        k += 1


def coprimesmod3(d,mod):
    k = 1
    while k <= d:
        if math.gcd(k,d) == 1 and ((k % 3) == mod):
            yield k
        k += 1


def coprimesaug(d,p,a):
    k = 1
    while k < d:
        if math.gcd(k,d) == 1 and k % p == a:
            yield k
        k += 1



# When a = 1, by taking p large enough we can make any choice of q_i b_i
↪  possible.
# Instead we are interested in fixing a prime p and then determining what q_i b_i
↪  are possible.
```

```
# The following code does this for some fixed prime p.

def finda1(p):

    primes = get_primes(p)

    result = []

    maxb = 1
    while pow(2, maxb - 2) < 2*p + 1:
        maxb += 1

    for b in range(1,maxb):
        for q in primes:
            # If the result is too large, there is nothing to check
            if pow(q,b) - 2*pow(q,b-1) + pow(q, b-2) < (2*p + 1)*2:
                result.append((pow(q,b) - 2*pow(q,b-1) + pow(q, b-2), q, b))
                 ↪  #(result, q, b)

    maxs = 0
    product = 1
    while product < 2*p + 1:
        product *= result[maxs-1][0]
        maxs += 1

    print('s='+str(maxs))
```

```
    r_values = []

    for prod in myproduct(result, maxs, p):
        if prod[0] <= 2*p:
            print('q='+str(prod[1])+'    b='+str(prod[2]))
            r = p
            for q in prod[1]:
                r *= pow(q, prod[2][prod[1].index(q)])
            r_values.append(r)
        elif prod[0] < 2*p + 1:
            fudge_factor = 1
            for q in prod[1]:
                fudge_factor *= (1-(1/q))
            if prod[0] <= 2*p + fudge_factor:
                print('q='+str(prod[1])+'    b='+str(prod[2]))
                r = p
                for q in prod[1]:
                    r *= pow(q, prod[2][prod[1].index(q)])
                r_values.append(r)

    return r_values


# Fix a prime p and find all values of r where elementary methods cannot show
↪  that the
# Reid-Tai sum is greater than 1. We then check all possible k_2 sums in order to
↪  find mc(r).
```

```
# If mc(r) >= 1, then the case is fine and the function does not return this r
 ↪  value.
# The r values returned by this function are the ones that may have bad Reid-Tai
 ↪  sums.

def computemcr(p):
    rvalues = finda1(p)

    # For the a = 2 and a = 3 cases, the additional r values to check depend on
     ↪  the value of p
    if p == 2:
        rvalues.append(8) # a = 3, p = 2
        rvalues.extend([4, 6]) # a = 2, p = 2
    elif p == 3:
        rvalues.append(54) # a = 3, p = 3, q = 2, b = 1
        rvalues.extend([9, 18, 36, 72, 90]) # a = 2, p = 3
    elif p == 5:
        rvalues.extend([25, 50, 100, 200, 75, 150, 450, 300]) # a = 2, p = 5
    else:
        ps = p*p
        rvalues.extend([ps, ps*2, ps*4, ps*8, ps*3, ps*6, ps*18, ps*12, ps*10]) #
         ↪  all a = 2 cases

    for r in rvalues:
        mc_big = True

        for k_1 in range(1, r):
```

```
        if mc_big and math.gcd(k_1, r) == 1:
            k_2 = r - k_1
            numsum = 0      # sum of numerators of fractional part
            for k_i in range(1, r):
                if math.gcd(k_i, r) == 1 and k_i % p == k_1 % p and k_1 !=
                ↪  k_i:
                    numsum += (k_2 + k_i) % r
            if numsum < r: # If the sum of the numerators is < r, then the
            ↪  fracsum is < 1
                print('mc('+str(r)+') < 1')
                mc_big = False

    if mc_big:
        print('mc('+str(r)+') >= 1')



# a generator for finding all possible combinations of primes and powers

def myproduct(args, s, p):

    result = [(arg[0], {(arg[1], arg[2])}) for arg in args]

    for prod in result:
        converted = prodconvert({}, tuple(prod[1])[0][0], tuple(prod[1])[0][1])
        yield (prod[0], tuple(converted[0]), tuple(converted[1]))
```

```
    i = 0
    while i < s:
        for pool in args:
            for prod in result:
                if len(prod[1]) < s and pool[1] not in (q[0] for q in prod[1]):
                    should_add = True
                    new_set = prod[1].union({(pool[1], pool[2])})
                    if True not in [new_set == altprod[1] for altprod in result]:
                    ↪  #ensures no repeats, current bottleneck in the algorithm
                        check = pool[0]*prod[0]
                        if check < 2*p + 1 or (check < 2*(2*p + 1) and True not
                        ↪  in [pairs[0] == 2 for pairs in new_set]):
                            result.append((pool[0]*prod[0],
                            ↪  prod[1].union({(pool[1], pool[2])})))
                            converted = prodconvert(prod[1], pool[1], pool[2])
                            yield (pool[0]*prod[0], tuple(converted[0]),
                            ↪  tuple(converted[1]))
        i += 1

# formats data in finda1
def prodconvert(prod, q, b):
    tup = tuple(prod)
    result1 = []
    result2 = []
    for entry in tup:
        result1.append(entry[0])
        result1.sort()
```

```
        result2.insert(result1.index(entry[0]), entry[1])  # sorts the q values
         ↪  by order and ensures that the b values stay paired correctly
    result1.append(q)
    result1.sort()
    result2.insert(result1.index(q), b)
    return [result1, result2]



# gathers candidate primes to check for a given p with a = 1
def get_primes(p):
    primes = []
    for q in gen_primes():
        if q-2 < (2*p + 1)*2:
            if q != p:
                primes.append(q)
        else:
            return primes


# a test function for computing mc(r)
def boundsum(r, p):
    minsum = 0
    minl = 0
    for l in range(1, p):
        #print('l='+str(l))
        if math.gcd(l, r) == 1:
            sum_l = 0
            for k_i in range(1,r):
```

```
                #print('k_i='+str(k_i))
                if k_i != l and math.gcd(k_i,r) == 1 and k_i % p == 1:
                    sum_l += k_i - l
                #print('sum_l='+str(sum_l))
            if minsum == 0 or sum_l < minsum:
                minsum = sum_l
                minl = l
    return (minl, minsum/r)

# Euler totient function (elementary implementation)
def phi(n):

    result = 1
    for i in range(2, n):
        if (math.gcd(i, n) == 1):
            result+=1
    return result

# Completes the first computation for the proof of canonical singularities away
↪ from ramification for p=3
def find_bad_d(rs):
    # How to eliminate d = 14, ..., 40
    #for d in [14,16,20,22,26,28,40]:
     #   p = phi(d)
      #  print((p*p + p)/(2*d))
    for d in [1,2,3,4,5,6,7,8,9,10,12,14,15,
    16,18,20,22,24,26,28,30,36,40,42,48,54,
```

```
60,66,84,90]:
    minsumd = 0
    firstsum = True
    for r in rs:
        for c in coprimes(r):
            if math.gcd(3,d) == 3:
                for a in [1,2]:
                    subresult = 0
                    for k_i in coprimesaug(d,3,a):
                        subresult += (d*c + k_i*r) % (r*d)
                    subresult = subresult / (r*d)
                    print('r = '+str(r)+", d = "+str(d)+", a = "+str(a)+", c
                    ↪  = "+str(c)+", age = "+str(subresult))
                    if subresult < minsumd or firstsum:
                        minsumd = subresult
                        firstsum = False
            else:
                subresult = 0
                for k_i in coprimes(d):
                    subresult += (d*c + k_i*r) % (r*d)
                subresult = subresult / (r*d)
                print('r = '+str(r)+", d = "+str(d)+", a = N/A, c =
                ↪  "+str(c)+", age = "+str(subresult))
                if subresult < minsumd or firstsum:
                    minsumd = subresult
                    firstsum = False
    #print(minsumd)
```

```
        if minsumd < 1:
            print('d='+str(d)+' has minsum = '+str(minsumd)+' < 1')
        else:
            print('d='+str(d)+' is looking good :)')




def prop3_7check():
    for r in [4,8,10]:
        minsum = 0
        firstsum = True
        for k_2 in coprimes(r):
            result = 0
            for k_i in coprimes(r):
                result += (k_2 + k_i) % r
            if result < minsum or firstsum:
                minsum = result
                firstsum = False

        print('r= '+str(r)+' has minsum '+str(minsum))



# Checks RT contribution for given e1, e2 acting on W and eigenvalues remaining
↪ from rep
def checkcontribution2n(e1, r1, e2, r2, eigenvalues):
    RT = 0
```

```
    for e in eigenvalues:
        RT += ((((r1-e1)*e[1]) + (e[0]*r1)) % (r1*e[1]))/(r1*e[1])
        RT += ((((r2-e2)*e[1]) + (e[0]*r2)) % (r2*e[1]))/(r2*e[1])
    if RT >= 1:
        print('(r1,r2) = ('+str(r1)+', '+str(r2)+') with eigenvalues
         ↪  ('+str(e1)+', '+str(e2)+') looks good!')
    else:
        print('contribution for (r1,r2) = ('+str(r1)+', '+str(r2)+') with
         ↪  eigenvalues ('+str(e1)+', '+str(e2)+') is '+str(RT))

def checkcontribution2nTF(e1, r1, e2, r2, eigenvalues):
    RT = 0
    print(eigenvalues)
    for e in eigenvalues:
        RT += ((((r1-e1)*e[1]) + (e[0]*r1)) % (r1*e[1]))/(r1*e[1])
        RT += ((((r2-e2)*e[1]) + (e[0]*r2)) % (r2*e[1]))/(r2*e[1])
    if RT >= 1:
        return False
    else:
        return True

def checkcontribution2nd(e1, r1, e2, r2, eigenvalues, d):
    RT = 0

    for e in eigenvalues:
        RT += ((((r1-e1)*e[1]) + (e[0]*r1)) % (r1*e[1]))/(r1*e[1])
        RT += ((((r2-e2)*e[1]) + (e[0]*r2)) % (r2*e[1]))/(r2*e[1])
```

```
    print('|   '+str(d)+'    |    '+str(RT)+'   |')


def checkcontribution2ndr(e1,r1,e2,eigenvalues,d):
    RT = 0


    for e in eigenvalues:
        RT += ((((r1-e1)*e[1]) + (e[0]*r1)) % (r1*e[1]))/(r1*e[1])
        RT += ((((r1-e2)*e[1]) + (e[0]*r1)) % (r1*e[1]))/(r1*e[1])
    print('|   '+str(d)+'    |    '+str(RT)+'   |')



################################################################
# START D=-3 CASE ##############################################
################################################################


#For the case D=-3, checks all possible contributions from V_r^{(k)} when W
↪  subset V_r^{(k)}
def mcr2nEisenstein(r):


    if r % 3 != 0:
        for e1 in coprimes(r):
            for e2 in coprimes(r):
                if e1 < e2:
                    RT = 0
                    for e in coprimes(r):
                        if e != e1 and e != e2:
                            RT += ((e-e1) % r)/r
```

```
                    RT += ((e-e2) % r)/r
                if RT < 1:
                    print('r = '+str(r)+' e1 = '+str(e1)+' e2 = '+str(e2)+'
                     ↪  RT = '+str(RT))
else:
    for e1 in coprimesmod3(r,1):
        for e2 in coprimesmod3(r,1):
            if e1 < e2:
                RT = 0
                for e in coprimesmod3(r,1):
                    if e != e1 and e != e2:
                        RT += ((e-e1) % r)/r
                        RT += ((e-e2) % r)/r
                if RT < 1:
                    print('r = '+str(r)+' e1 = '+str(e1)+' e2 = '+str(e2)+'
                     ↪  RT = '+str(RT))
    for e1 in coprimesmod3(r,2):
        for e2 in coprimesmod3(r,2):
            if e1 < e2:
                RT = 0
                for e in coprimesmod3(r,2):
                    if e != e1 and e != e2:
                        RT += ((e-e1) % r)/r
                        RT += ((e-e2) % r)/r
                if RT < 1:
                    print('r = '+str(r)+' e1 = '+str(e1)+' e2 = '+str(e2)+'
                     ↪  RT = '+str(RT))
```

```
def checkall2n():
    for r1 in [1,2,3,4,6,12]:
        for r2 in [1,2,3,4,6,12]:
            if r1 <= r2:
                for e1 in coprimes(r1):
                    for e2 in coprimes(r2):
                        eigenvalues = []

                        # Branch r1 is not divisible by 3
                        if r1 % 3 != 0:
                            for e in coprimes(r1):
                                if e != e1:
                                    eigenvalues.append([e,r1])
                            # r1 and r2 not divisible
                            if r2 % 3 != 0:
                                for e in coprimes(r2):
                                    if e != e2:
                                        eigenvalues.append([e,r2])
                                checkcontribution2n(e1,r1,e2,r2,eigenvalues)
                            # r1 not divisible r2 is
                            else:
                                for e in coprimesmod3(r2,e2 % 3):
                                    if e != e2:
                                        eigenvalues.append([e,r2])
                                checkcontribution2n(e1,r1,e2,r2,eigenvalues)
```

```
                        # r1 is divisible by 3
                        else:
                            for e in coprimesmod3(r1, e1 % 3):
                                if e != e1:
                                    eigenvalues.append([e,r1])
                            # r1 is divisible and r2 not divisible
                            if r2 % 3 != 0:
                                for e in coprimes(r2):
                                    if e != e2:
                                        eigenvalues.append([e,r2])
                                checkcontribution2n(e1,r1,e2,r2,eigenvalues)
                            # r1 and r2 both divisible
                            else:
                                for e in coprimesmod3(r2,e2 % 3):
                                    if e != e2:
                                        eigenvalues.append([e,r2])
                                checkcontribution2n(e1,r1,e2,r2,eigenvalues)


def getalleigen2n():
    result = []
    for r1 in [1,2,3,4,6,12]:
        for r2 in [1,2,3,4,6,12]:
            if r1 <= r2:
                for e1 in coprimes(r1):
                    for e2 in coprimes(r2):
                        eigenvalues = []
```

```
# Branch r1 is not divisible by 3
if r1 % 3 != 0:
    for e in coprimes(r1):
        if e != e1:
            eigenvalues.append([e,r1])
    # r1 and r2 not divisible
    if r2 % 3 != 0:
        for e in coprimes(r2):
            if e != e2:
                eigenvalues.append([e,r2])
        if checkcontribution2nTF(e1,
        r1,
        e2,
        r2,
        eigenvalues):
             result.append([e1,r1,e2,r2])
    # r1 not divisible r2 is
    else:
        for e in coprimesmod3(r2,e2 % 3):
            if e != e2:
                eigenvalues.append([e,r2])
        if checkcontribution2nTF(e1,
        r1,
        e2,
        r2,
        eigenvalues):
             result.append([e1,r1,e2,r2])
```

```
# r1 is divisible by 3
else:
    for e in coprimesmod3(r1, e1 % 3):
        if e != e1:
            eigenvalues.append([e,r1])
    # r1 is divisible and r2 not divisible
    if r2 % 3 != 0:
        for e in coprimes(r2):
            if e != e2:
                eigenvalues.append([e,r2])
        if checkcontribution2nTF(e1,
        r1,
        e2,
        r2,
        eigenvalues):
             result.append([e1,r1,e2,r2])
    # r1 and r2 both divisible
    else:
        for e in coprimesmod3(r2,e2 % 3):
            if e != e2:
                eigenvalues.append([e,r2])
        if checkcontribution2nTF(e1,
        r1,
        e2,
        r2,
        eigenvalues):
             result.append([e1,r1,e2,r2])
```

```
    return result


# Check all d in Eisenstein case
def checkalld(e1, r1, e2, r2):
    for d in [1,2,3,4,6,12]:
        eigenvalues = []
        if d % 3 == 0:
            for e in coprimesmod3(d, 1):
                eigenvalues.append([e, d])
            checkcontribution2nd(e1,r1,e2,r2,eigenvalues,d)
            eigenvalues = []
            for e in coprimesmod3(d,2):
                eigenvalues.append([e,d])
            checkcontribution2nd(e1,r1,e2,r2,eigenvalues,d)
        else:
            for e in coprimes(d):
                eigenvalues.append([e, d])
            checkcontribution2nd(e1,r1,e2,r2,eigenvalues,d)




# Prints table of values for Eisenstein case and all possible combos of r_1 r_2.
def checkdforallr():
     ers = getalleigen2n()
     #print(ers)

     for data in ers:
```

```
        #OLD FORMATTING
        print('|-----------------------------------|')
        print('|  (r1,r2)=('+str(data[1])+','+str(data[3])+')
         ↪  (e1,e2)=('+str(data[0])+','+str(data[2])+')   |')
        print('| ---------------------------------- |')
        print('|   d    |       Contributions       |')
        checkalld(data[0],data[1],data[2],data[3])



# Checks contributions of different r2 for fixed r1=7. Ultimately shows r1=7
 ↪  implies sum(g) geq 1
def checkcontributionsgenball():
     for r2 in [1,2,3,4,6,7,12,14]:
          for a in coprimes(r2):
               contribution = (((2*r2) + (7*a))%(7*r2)) / (7*r2)
               contribution += (((4*r2) + (7*a))%(7*r2)) / (7*r2)
               if contribution < (3/7):
                    print('r2 = '+str(r2)+' a = '+str(a)+' has contribution
                     ↪  '+str(contribution)+' in case 1!')
               contribution = (((5*r2) + (7*a))%(7*r2)) / (7*r2)
               contribution += (((6*r2) + (7*a))%(7*r2)) / (7*r2)
               if contribution < (2/7):
                    print('r2 = '+str(r2)+' a = '+str(a)+' has contribution
                     ↪  '+str(contribution)+' in case 2!')
```

B.2 Macaulay2 Code

```
-- Run at startup
```

```
loadPackage "NormalToricVarieties"


-- Computing a resolution for the cone over the rational normal curve (1/3(1,1))
rayList = {{3,-1},{0,1}}
coneList = {{0,1}}
X = normalToricVariety(rayList, coneList)
resX = makeSmooth X
rays resX
max resX


-- Computing a resolution for the threefold singularity of type 1/3(1,1,0)
rayList = {{3,-1,0},{0,1,0},{0,0,1}}
coneList = {{0,1,2}}
X = normalToricVariety(rayList, coneList)
resX = makeSmooth X
rays resX
max resX


-- Computing a resolution for the sevenfold singularity of type
↪ 1/3(1,1,0,0,0,0,0)
rayList = {{3,-1,0,0,0,0,0}, {0,1,0,0,0,0,0},{0,0,1,0,0,0,0}, {0,0,0,1,0,0,0},
↪ {0,0,0,0,1,0,0},{0,0,0,0,0,1,0}, {0,0,0,0,0,0,1}}
coneList = {{0,1,2,3,4,5,6}}
X = normalToricVariety(rayList, coneList)
resX = makeSmooth X
rays resX
max resX
```

```
-- Computing a resolution for the surface singularity 1/6(1,1)

rayList = {{6,-1},{0,1}}

coneList = {{0,1}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for the threefold singularity of type 1/6(1,1,0)

rayList = {{6,-1,0},{0,1,0},{0,0,1}}

coneList = {{0,1,2}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for the threefold singularity of type 1/6(1,1,1)

rayList = {{6,-1,-1},{0,1,0},{0,0,1}}

coneList = {{0,1,2}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/6(1,1,2)

-- Note, this is done in Dawes)
```

```
rayList = {{6,-1,-2},{0,1,0},{0,0,1}}

coneList = {{0,1,2}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/6(1,1,3)

rayList = {{6,-1,-3},{0,1,0},{0,0,1}}

coneList = {{0,1,2}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/6(1,1,1,1)

rayList = {{6,-1,-1,-1},{0,1,0,0},{0,0,1,0},{0,0,0,1}}

coneList = {{0,1,2,3}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/6(1,1,1,2)

rayList = {{6,-1,-1,-2},{0,1,0,0},{0,0,1,0},{0,0,0,1}}

coneList = {{0,1,2,3}}

X = normalToricVariety(rayList, coneList)
```

```
resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/12(1,2,7)

rayList = {{12,-2,-7},{0,1,0},{0,0,1}}

coneList = {{0,1,2}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX


-- Computing a resolution for type 1/12(1,2,1,7)

rayList = {{12,-2,-1,-7},{0,1,0,0},{0,0,1,0},{0,0,0,1}}

coneList = {{0,1,2,3}}

X = normalToricVariety(rayList, coneList)

resX = makeSmooth X

rays resX

max resX
```