

MATH 4230/5230: Differential Geometry of Curves and Surfaces

Quiz 8

Friday, October 23, 2026

Name: _____

Score

____/10

Problem. Give the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = -x^2 + y^2\}$$

its upward orientation. Compute the Weingarten map $\mathcal{W}_p : T_p S \rightarrow T_p S$ at $p = (0, 0, 0)$. Show your work.

MATH 4230/5230 — Quiz 8 Solutions

Parametrize S by

$$\sigma : \mathbb{R}^2 \longrightarrow S, \quad \sigma(x_1, x_2) = (x_1, x_2, -x_1^2 + x_2^2).$$

Its Jacobian matrix is

$$J(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2x_1 & 2x_2 \end{pmatrix}.$$

At $(0, 0)$, the columns are $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$, so $T_p S = \text{span}\{e_1, e_2\}$. In particular,

$$(T_{(0,0)}\sigma)((1, 0)) = e_1, \quad (T_{(0,0)}\sigma)((0, 1)) = e_2.$$

Taking the cross product of the columns of $J(\sigma)$ and normalizing gives the upward-pointing Gauss map:

$$(N \circ \sigma)(x_1, x_2) = \frac{(2x_1, -2x_2, 1)}{\sqrt{1 + 4x_1^2 + 4x_2^2}}.$$

At $(0, 0)$, the denominator has value 1, and both of its first partial derivatives vanish. Therefore,

$$\frac{\partial(N \circ \sigma)}{\partial x_1}(0, 0) = (2, 0, 0), \quad \frac{\partial(N \circ \sigma)}{\partial x_2}(0, 0) = (0, -2, 0).$$

Equivalently,

$$J(N \circ \sigma)|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}.$$

The chain rule gives

$$T_{(0,0)}(N \circ \sigma) = (T_p N) \circ T_{(0,0)}\sigma,$$

so

$$T_p N(e_1) = T_p N((T_{(0,0)}\sigma)((1, 0))) = T_{(0,0)}(N \circ \sigma) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2e_1$$

and, similarly, $T_p N(e_2) = -2e_2$. Hence $\mathcal{W}_p = -T_p N$ has matrix form in the standard coordinates as

$$[\mathcal{W}_p]_{(e_1, e_2)} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

In other words, for all $(a, b, 0) \in T_p S$, we have $\mathcal{W}_p(a, b, 0) = (-2a, 2b, 0)$.

We have also seen that for the graph of a function $f(x_1, x_2)$ at a critical point p , we have

$$\mathcal{W}_p = \begin{bmatrix} f_{x_1 x_1}(p) & f_{x_1 x_2}(p) \\ f_{x_2 x_1}(p) & f_{x_2 x_2}(p) \end{bmatrix}$$

which also gives the result above.