Midterm 2

Linear Algebra: Matrix Methods

MATH 2130

Fall 2022

Friday October 28, 2022

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NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 45 minutes to complete the exam.

1. • *Compute the determinant of each of the following matrices:*

(a) (10 points) $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

SOLUTION:

Solution. We have $\boxed{\det A = -1}$ The fastest way to see this may be to expand off of the third column (or even to interchange two columns, twice); however, to use the standard method, we have

$$\det A = (4)[(-2)(0) - (0)(1)] - (-1)[(-1)(0) - (0)(0)] + (1)[(-1)(1) - (-2)(0)] = -1.$$

□

(b) (10 points) $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}$

SOLUTION:

Solution. We have $\boxed{\det B = -2}$ We use row operations:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \pi \\ 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 5 & 1 & 0 & 2 & 10^4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 0 & e & -4 & 8 & 3^{-5} \\ 0 & 1 & 0 & 0 & 0 & \pi \\ 0 & 0 & 1 & 0 & 2 & 10^4 - 5\pi \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -2$$

$\square$

1
20 points

2. • Let V be a real vector space and let $W_1, W_2 \subseteq V$ be two real subvector spaces of V. We define the *union* $W_1 \cup W_2$ of W_1 and W_2 to be the collection of vectors in V that are in W_1 or W_2:

$$W_1 \cup W_2 := \{\mathbf{v} \in V : \mathbf{v} \in W_1 \text{ or } \mathbf{v} \in W_2\}.$$

We define the *intersection* $W_1 \cap W_2$ of W_1 and W_2 to be the collection of vectors in V that are in W_1 and W_2:

$$W_1 \cap W_2 := \{\mathbf{v} \in V : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}.$$

(a) (10 points) **True** or **False**: *If $W_1, W_2 \subseteq V$ are two real subvector spaces of a real vector space V, then the union $W_1 \cup W_2$ is a real subvector space of V.*

If true, provide a proof. If false, provide an example and prove that the example shows the statement is false. Your solution must start with the sentence "This statement is TRUE" or "This statement is FALSE".

SOLUTION:

Proof. This statement is FALSE. For example, let $V = \mathbb{R}^2$, let $W_1 = \{(x_1, 0) \in \mathbb{R}^2\}$ be the x_1-axis and let $W_2 = \{(0, x_2) \in \mathbb{R}^2\}$ be the x_2-axis. The lines W_1 and W_2 are both real subvector spaces of $\mathbb{R}^2$, but the union $W_1 \cup W_2$ of the two lines is not a real subvector space of $\mathbb{R}^2$ since $(1, 0) \in W_1 \cup W_2$ (since $(1, 0) \in W_1$) and $(0, 1) \in W_1 \cup W_2$ (since $(0, 1) \in W_2$), but the vector $(1, 0) + (0, 1) = (1, 1)$ is not in $W_1 \cup W_2$ since it is in neither W_1 nor W_2.

$\square$

(b) (10 points) **True** or **False**: *If $W_1, W_2 \subseteq V$ are two real subvector spaces of a real vector space V, then the intersection $W_1 \cap W_2$ is a real subvector space of V.*

If true, provide a proof. If false, provide an example and prove that the example shows the statement is false. Your solution must start with the sentence "This statement is TRUE" or "This statement is FALSE".

SOLUTION:

Proof. This statement is TRUE. We must show that $W_1 \cap W_2$ is non-empty, and closed under addition and scaling. We have $0 \in W_1 \cap W_2$ since $0 \in W_1$ and $0 \in W_2$, so $W_1 \cap W_2$ is not empty. Now, if $\mathbf{u}, \mathbf{v} \in W_1 \cap W_2$, then we have $\mathbf{u} + \mathbf{v} \in W_1 \cap W_2$ since $\mathbf{u} + \mathbf{v} \in W_1$ and $\mathbf{u} + \mathbf{v} \in W_2$ (W_1 and W_2 are closed under addition). Therefore $W_1 \cap W_2$ is closed under addition. If $\mathbf{v} \in W_1 \cap W_2$ and $\lambda \in \mathbb{R}$, then $\lambda\mathbf{v} \in W_1 \cap W_2$ since $\lambda\mathbf{v} \in W_1$ and $\lambda\mathbf{v}$ in W_2 (W_1 and W_2 are closed under scaling). Therefore, $W_1 \cap W_2$ is closed under scaling. $\square$

2
20 points

3. (20 points) • Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be bases for a real vector space V, and suppose that

$$\begin{aligned} \mathbf{v}_1 &= 4\mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3 \\ \mathbf{v}_2 &= 3\mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3 \\ \mathbf{v}_3 &= 7\mathbf{w}_1 + 23\mathbf{w}_2 - 2\mathbf{w}_3 \end{aligned}$$

Find the change-of-coordinates matrix to go from the coordinates with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *to the coordinates with respect to the basis* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

SOLUTION:

Solution. The change-of-coordinates matrix to go from the coordinates with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the coordinates with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$can be read off from the equations above as the matrix

$$\begin{bmatrix} 4 & -1 & 1 \\ 3 & 2 & -1 \\ 7 & 23 & -2 \end{bmatrix}^T = \boxed{\begin{bmatrix} 4 & 3 & 7 \\ -1 & 2 & 23 \\ 1 & -1 & -2 \end{bmatrix}}.$$

$\square$

3
20 points

4. (20 points) • Find a basis for the solution space of the difference equation

$$y_{k+2} + y_{k+1} - 56y_k = 0.$$

SOLUTION:

Solution. A basis for the solution space of the difference equation $y_{k+2} + y_{k+1} - 56y_k = 0$ is given by the sequences

$$\boxed{\{y_k\} = \{(-8)^k\} \quad \text{and} \quad \{y_k\} = \{7^k\}}.$$

To find this we consider the auxiliary equation

$$z^2 + z - 56 = 0.$$

Since this factors as $z^2 + z - 56 = (z+8)(z-7) = 0$, we see that the solutions are $z = -8$ and $z = 7$. We have seen in class, and in the book, that since this is a degree 2 difference equation, with two distinct solutions to the auxiliary equation, that $\{y_k\} = \{(-8)^k\}$ and $\{y_k\} = \{7^k\}$ form a basis for the solution space. □

4
20 points

5. • Consider the following real matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

(a) (5 points) *Find the characteristic polynomial $p_A(t)$ of A.*

SOLUTION:

Solution. We have

$$\begin{aligned}
p_A(t) &= \begin{vmatrix} t-2 & 1 & -1 \\ 0 & t-3 & 1 \\ -2 & -1 & t-3 \end{vmatrix} \\
&= (t-2)[(t-3)^2 - (1)(-1)] - (1)[0 - (1)(-2)] + (-1)[0 - (t-3)(-2)] \\
&= (t-2)[t^2 - 6t + 10] - 2 + \underbrace{(t-3)(-2)}_{-2t+6} \\
&= (t^3 - 6t^t + 10t - 2t^2 + 12t - 20) - 2 + (6 - 2t) \\
&= t^3 - 8t^2 + 20t - 16.
\end{aligned}$$

In other words, the solution is:

$$\boxed{p_A(t) = t^3 - 8t^2 + 20t - 16.}$$

□

As a quick partial check of the solution, observe that

$$\begin{aligned}
\operatorname{tr}(A) &= 8 \\
\det A &= \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2(6+2) = 16.
\end{aligned}$$

confirming the computation of the coefficients of t^2 and t^0, since we know that

$$p_A(t) = t^3 - \operatorname{tr}(A)t^2 + \alpha t + (-1)^3 \det(A)$$

for some real number $\alpha \in \mathbb{R}$.

(b) (5 points) *Find the eigenvalues of A.*

SOLUTION:

Solution. One can easily check that

$$p_A(2) = 2^3 - 8 \cdot 2^2 + 20 \cdot 2 - 16 = 8 - 32 + 40 - 16 = 48 - 48 = 0.$$

Thus $(t-2)$ is a factor of $p_A(t)$, so that we have

$$p_A(t) = (t-2)(t^2 - 6t + 8) = (t-2)(t-2)(t-4).$$

Thus the eigenvalues are

$$\boxed{\lambda = 2, 4.}$$

□

(c) (5 points) *Find a basis for each eigenspace of A in $\mathbb{R}^3$.*

SOLUTION:

Solution. To find a basis for the $\lambda = 2$ eigenspace E_2, we compute

$$E_2 := \ker(2I - A) = \ker \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We add rows, and get the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_2 = \left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Now we compute a basis for the $\lambda = 4$ eigenspace E_4. We have

$$E_4 = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus we have

$$E_4 = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Thus the solution to the problem is:

The eigenspaces for A are E_2 and E_4, and we have that $\boxed{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$ is a basis for E_2 and $\boxed{\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$ is a basis for E_4. $\square$

(d) (5 points) *Is A diagonalizable? If so, find a matrix $S \in \mathrm{M}_{3\times 3}(\mathbb{R})$ so that $S^{-1}AS$ is diagonal. If not, explain.*

SOLUTION:

Solution. $\boxed{\text{No.}}$ A is not diagonalizable since we showed in part (c) that there does not exists a basis of $\mathbb{R}^3$ consisting of eigenvectors for A. □

5
20 points